6.S979 Lecture 9:

Reminder: Pset 1 was due on 10/2

Last time:

- Correlation sets
- XOR correlations

$$p(a,b|x,y)$$

$$C_{xy} = \sum_{a,b} a \cdot b \; p(a,b|x,y)$$

$$= \langle \psi | A_x \otimes B_y | \psi \rangle$$

$$C_{xx'} = \langle \psi | A_x A_{x'} | \psi \rangle$$

$$C_{yy'} = \langle \psi | B_y B_{y'} | \psi \rangle$$

$x \qquad y$

$$C = \begin{array}{c} x\{ \\ \\ y\{ \end{array} \begin{pmatrix} \ddots & & C_{xy} \\ & \ddots & \\ C_{yx} = C_{xy} & & \ddots \end{pmatrix}$$

<u>Conditions:</u>

1) C is Hermitian $(C = C^{\dagger})$

$C_{xx} = C_{yy} = I$

2) $C_{xy} = C_{yx} \Rightarrow C_{xy} \in \mathbb{R}$

3) $|C_{xy}| \leq 1$

4) $C \succeq 0$

$C_{ij} = \langle u_i, u_j \rangle \Rightarrow C \succeq 0$

$$u_i = A_x|\psi\rangle \text{ or } B_y|\psi\rangle$$

$$\begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & C_{xy} \\ C_{xy} & 1 \end{pmatrix} \succeq 0$$

Suppose $C_{xy} > 1$

$$(1 \ -1) \begin{pmatrix} 1 & C_{xy} \\ C_{xy} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 2 - 2C_{xy} < 0$$

Thm [Tsirelson '87]
These criteria completely characterize valid C matrices

<u>Pf:</u> Given valid C,
construct $|\psi\rangle$, A_x, B_y
s.t. $C_{xy} = \langle\psi| A_x B_y |\psi\rangle$

1) $C \succeq 0 \Rightarrow C$ is a gram matrix

$$C = \sum_{k=1}^{r \leftarrow \mathrm{rank}(C)} \lambda_k \underbrace{\vec{u}_k \vec{u}_k^{\dagger}}$$

λ_k: k-th eigenvalue w/ multiplicity

$\vec{u}_k \vec{u}_k^{\dagger}$: projector onto k-th eigenspace

$\lambda_k > 0$

$$C_{ij} = \sum_{k=1}^{r} \lambda_k (\vec{u}_k)_i \overline{(\vec{u}_k)_j}$$

$$= \left\langle \underbrace{\begin{pmatrix} \sqrt{\lambda_1}\,\overline{(\vec{u}_1)_i} \\ \sqrt{\lambda_2}\,\overline{(\vec{u}_2)_i} \\ \vdots \\ \sqrt{\lambda_r}\,\overline{(\vec{u}_r)_i} \end{pmatrix}}_{\vec{v}_i \in \mathbb{C}^r}, \underbrace{\begin{pmatrix} \sqrt{\lambda_1}\,\overline{(\vec{u}_1)_j} \\ \sqrt{\lambda_2}\,(\vec{u}_2)_j \\ \vdots \\ \sqrt{\lambda_r}\,\overline{(\vec{u}_r)_j} \end{pmatrix}}_{\vec{v}_j} \right\rangle$$

$$\langle \vec{a}, \vec{b} \rangle = \sum_i \overline{a_i}\, b_i$$

$$C_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$$

$$\vec{w}_i = \big(\mathrm{Re}((\vec{v}_i)_1), \mathrm{Im}((\vec{v}_i)_1), \ldots, \mathrm{Re}((\vec{v}_i)_r), \mathrm{Im}((\vec{v}_i)_r) \big)$$

$\in \mathbb{R}^{2r}$

$$\langle \tilde{w}_i, \tilde{w}_j \rangle = \mathrm{Re} \langle \tilde{v}_i, \tilde{v}_j \rangle$$

<u>So far:</u>

Given $C \succeq 0 \Rightarrow \exists\, \tilde{w}_1, \dots, \tilde{w}_m \in \mathbb{R}^{2r}$

s.t. $\mathrm{Re}\, C_{ij} = \langle \tilde{w}_i, \tilde{w}_j \rangle$

<u>Next:</u> Construct a q. strat

$$|\psi\rangle = |EPR\rangle^{\otimes n}$$

TBD

To construct observables, define "<u>Clifford</u> algebra" ambiguous

Observables on n qubits $(\mathbb{C}^{2^n})$

$T_1, \ldots, T_{2n}$ $\qquad T_i^2 = I$

$T_i T_j = -T_j T_i \quad \forall\, i \neq j$

"pairwise anticommuting"

$T_1 = X \otimes I \otimes \ldots \otimes I$

$T_2 = Z \otimes I \otimes \ldots \otimes I$

$T_3 = Y \otimes X \otimes I \otimes \ldots \otimes I$

$T_4 = Y \otimes Z \otimes I \otimes \ldots \otimes I$

$T_5 = Y \otimes Y \otimes X \otimes I \ldots \otimes I$

$T_6 = Y \otimes Y \otimes Z \otimes I \ldots \otimes I$

Given $\tilde{w}_x, \tilde{w}_y \in \mathbb{R}^{2r}$

$n = r$

$$A_x = \left(\sum_{k=1}^{2r} (\tilde{w}_x)_k \cdot \bar{T}_k \right)_A \otimes I_B$$

$$B_y = I_A \otimes \left(\sum_{k=1}^{2r} (\tilde{w}_y)_k \cdot \bar{T}_k \right)$$

$$A_x^2 = \sum_{k=1}^{2r} (\tilde{w}_x)_k^2 \cdot \bar{T}_k^2 = I$$

$$+ \cancelto{0}{\sum_{k \neq k'} (\tilde{w}_x)_k (\tilde{w}_x)_{k'} \left(\bar{T}_k \cdot \bar{T}_{k'} + \bar{T}_{k'} \cdot \bar{T}_k \right)}$$

$$= \sum_{k=1}^{2r} (\tilde{w}_x)_k^2 \cdot I = \langle \tilde{w}_x, \tilde{w}_x \rangle I$$

$$= C_{xx} \cdot I$$

$$= I$$

$\tilde{w}_i$'s are unit vectors

✓

$$C_{xy} \stackrel{?}{=} \langle \psi | A_x B_y | \psi \rangle$$

$$= \langle EPR^{\otimes n} | A_x B_y | EPR^{\otimes n} \rangle$$

$$= \frac{1}{2^n} \operatorname{tr}(A_x B_y^T)$$

$$= \frac{1}{2^n} \operatorname{tr}(\underset{\sim}{A_x} \underset{\sim}{\overline{B_y}})$$

$$= \frac{1}{2^n} \operatorname{tr}\left(\sum_{k,\ell=1}^{2r} (\tilde{w}_x)_k \cdot (\tilde{w}_y)_\ell \, T_k T_\ell \right)$$

$$= \frac{1}{2^n} \sum_{k,l=1}^{2n} (\tilde{w}_x)_k \cdot (\tilde{w}_y)_l \, \mathrm{tr}(T_k T_l)$$

$\mathrm{tr}(AB) = \mathrm{tr}(BA)$

If $AB = -BA$

then $\mathrm{tr}(AB) = 0$

$$= \cancel{\frac{1}{2^n}} \sum_k (\tilde{w}_x)_k \cdot (\tilde{w}_y)_k \cdot \cancel{\mathrm{tr}(I)}$$

$$= \sum_k (\tilde{w}_x)_k \cdot (\tilde{w}_y)_k$$

$$= \langle \tilde{w}_y, \tilde{w}_x \rangle$$

$$= C_{xy}$$

$\blacksquare$

Nice things:

1) Assumed only that C was a "commuting operator" correlation, on possibly infinite dim state

But showed $\exists\ |\psi\rangle, A_x, B_y$ realizing C w/ tensor product and $\dim_A = 2^n$

$$n = \text{rank}(C)$$
$$\leq \#\text{ Alice qb's} + \#\text{ Bob qb's}$$

"$C_{q}^{\oplus} = C_{qs}^{\oplus} = C_{qa}^{\oplus} = C_{qc}^{\oplus}$"

2) $\exists$ an algorithm to check if given XOR corr. C is realizable

"semidefinite program"

[Cleve, Høyer, Toner, Watrous '04]

3) XOR game

$$\beta^* = \max_{|\psi\rangle, A, B} \mathbb{E}_{x,y} G_{xy} \langle \psi | A_x B_y | \psi \rangle$$

"quantum entangled bias"

$$\beta^* = \max_{\|\vec{u}_x\| = \|\vec{u}_y\| = 1} \mathbb{E}_{x,y} G_{xy} \langle \vec{u}_x, \vec{u}_y \rangle$$

$$\beta = \max_{a_x, a_y \in \{\pm 1\}} \mathop{\mathbb{E}}_{x,y} G_{xy}(a_x \cdot a_y)$$

We saw for CHSH

$$\beta^* = \sqrt{2}\, \beta$$

Q: For a general XOR game, how big can $\frac{\beta^*}{\beta}$ be?

"bias ratio"

A: $\beta^*/\beta \leq O(1)$

Pf: Grothendieck's inequality

$\forall$ real M, $\vec{u}_i, \vec{v}_j$ unit vectors

$$\left| \sum_{i,j} M_{ij} \langle \vec{u}_i, \vec{v}_j \rangle \right|$$

$$\leq K_{\mathbb{R}}^{G} \max_{a_i, b_j \in \{\pm 1\}} \sum_{ij} M_{ij} a_i b_j$$

$K_{\mathbb{R}}^{G} \leq 1.783$

Note: Not true for 3 or more parties

- Approximate dimension bounds

We saw $\dim \leq 2^n = 2^{\#\text{questions}}$

Lemma [Johnson Lindenstrauss]

$v_1, \ldots, v_n \in \mathbb{R}^d$

$\exists$ mapping $f : \mathbb{R}^d \to \mathbb{R}^K$

$$(1-\varepsilon)\|v_i - v_j\|^2 \leq \|f(v_i) - f(v_j)\|_2^2$$

$$\leq (1+\varepsilon)\|v_i - v_j\|$$

$$K = \Omega\left(\frac{1}{\varepsilon^2}\log n\right)$$

independent of d

$\Rightarrow$ For any XOR correlation, you can ε-approximately realize w/ dim $2^{\frac{1}{\varepsilon^2}\log(\#\text{ questions})}$

[CHTW '04]

Limitations:

Doesn't work for more parties or non-XOR correlations

E.g. Say I'm given C_{xy}

and also $D_x = \langle \psi | A_x | \psi \rangle$

You might try

$$C = \begin{array}{c c} & \begin{array}{ccc} \phi & x & y \end{array} \\ \begin{array}{c} \phi \\ x \\ y \end{array} & \begin{pmatrix} 1 & D_x & D_y \\ D_x^{\dagger} & C_{xx} & C_{xy} \\ D_y^{\dagger} & C_{yx} & C_{yy} \end{pmatrix} \end{array}$$

$$C_{ij} = \langle \psi | A_i A_j | \psi \rangle$$

$$A_\phi = I$$

You'd want

$$\tilde{w}_x \longrightarrow A_x$$

to send

$$\tilde{w}_\phi \longrightarrow I$$

But this doesn't work

E.g. C_{xy} to be maximal CHSH correlations

$$\langle A_0 \rangle = 1$$

This expanded C matrix says these are consistent, <u>but</u>

Self-testing $\Rightarrow$ impossible