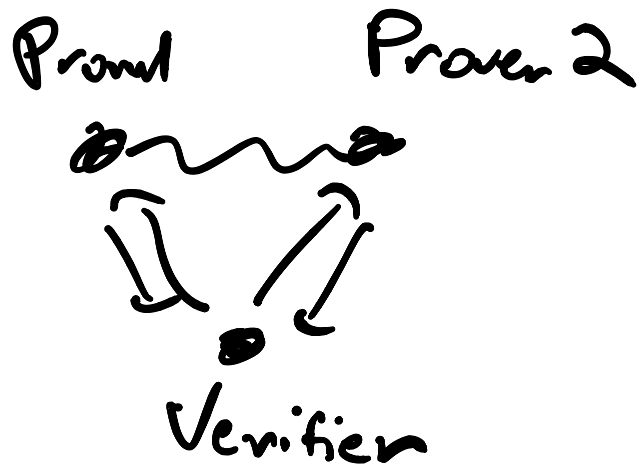
6.S979 Lecture 14

- Pset 2 out 11/6

Last time:



$$\mathrm{MIP}[\mathcal{C}]$$

$$\mathrm{MIP}[\mathcal{C}_{\text{classical}}] = \mathrm{MIP}$$

$$\mathrm{MIP}[\mathcal{C}_{qa}] = \mathrm{MIP}^*$$

$$\mathrm{MIP}[\mathcal{C}_{qc}] = \mathrm{MIP}^{co}$$

Obs 1:

$$\mathcal{C}_{\text{classical}}^{\oplus} \subsetneq \mathcal{C}_{q}^{\oplus}$$

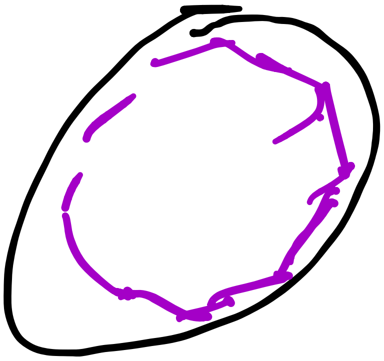
$$\mathrm{MIP}^{\oplus} = \mathrm{NEXP}$$

$$\mathrm{MIP}^{\oplus *} \subseteq \mathrm{EXP}$$

$P \neq NP \qquad EXP \neq NEXP$

$$\Updownarrow$$

$$MIP^{\oplus *} \subsetneq MIP^{\oplus}$$



Obs 2: Alg. for approximating

$$(*) \quad \max_{p \in C} \langle p, \vec{w} \rangle$$

$\Longrightarrow$ upper bound on $MIP[C]$

lower bounds on $MIP[C]$

$\Longrightarrow$ hardness for approx. $(*)$

If $C_{qa} = C_{qc}$

$\Rightarrow$ approx. (*) over C_{qa} is computable

$\Rightarrow$ MIP* only contains ~~dec.~~ computable languages

Def: The problem of approximating (*) is given $\tilde{v}$, decide whether

- YES: $\max_{p \in C} \langle p, \tilde{v} \rangle \geq 2/3$
- NO: $\max_{p \in C} \langle p, \tilde{v} \rangle \leq 1/3$

promised that one of two holds

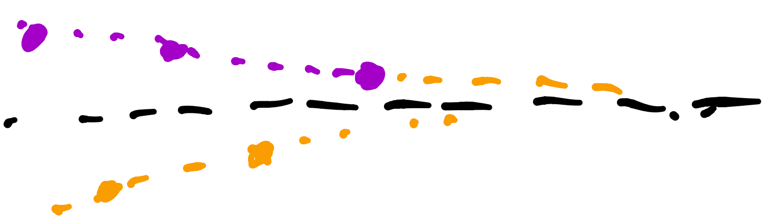
Recall

NPA hierarchy $\longrightarrow$ C_{qc}

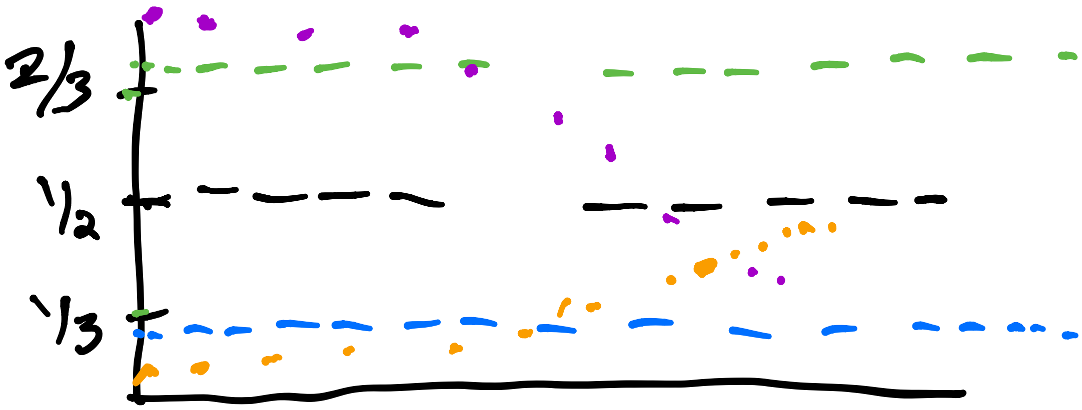
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

Fixed dimension $\longrightarrow$ C_{qa}

Alg: interleave computing max over NPA, inner approx.



true max



- $> 1/3 \Rightarrow$ YES
- $< 2/3 \Rightarrow$ NO

this algorithm always terminates
if promise holds

Consequence:

If MIP* contains an uncomputable language, then $C_{qa} \neq C_{qc}$

(nonconstructive) $\Rightarrow C_{qa} \neq C_{qc}$

MIP* = RE

"recursively enumerable"

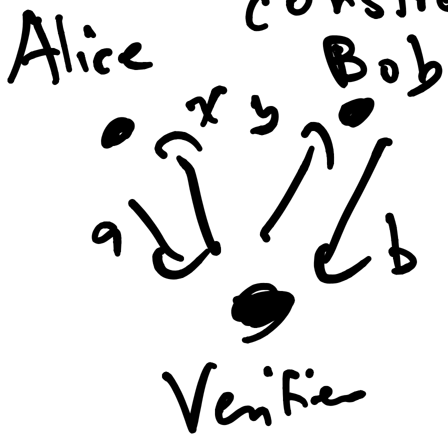
the halting problem

[Ji N Vidick Wright Yuen '20]

Plan:

- First explain MIP = NEXP
- Generalize those techniques to MIP*

MIP is closely connected to constraint satisfaction problems.



Roughly:

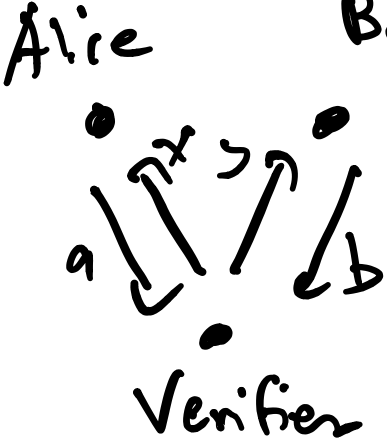
Given an MIP protocol, computing max acceptance prob

$\Updownarrow$ as hard as

Given a CSP, determining max # of constraints that can be satisfied

Graph coloring
3SAT $\longrightarrow$

1) MIP protocol $\longrightarrow$ CSP



$p(a, b | x, y)$

wlog, Alice & Bob are deterministic

Alice has an assignment

$$(a_{x_1}, a_{x_2}, \ldots, a_{x_k})$$

Bob has

$$(b_{y_1}, b_{y_2}, \ldots, b_{y_k})$$

$$a_{x_1}, a_{x_2}, \ldots, a_{x_k}, b_{y_1}, \ldots, b_{y_k}$$

$$V[(a_x, b_y)]$$

$= 1$ if a_x, b_y satisfy some condition

Prob. that V accepts on this strat.

(assume that V samples pairs $(x,y) \sim S$ uniformly)

S: set of pairs of questions

$=$ fraction of $(x,y) \in S$ s.t. (a_x, b_y) satisfy condition

this is a constraint satisfaction prob.

$$a_{x_1}, \ldots, a_{x_k}, b_{y_1}, \ldots, b_{y_k}$$

2) CSP $\longrightarrow$ MIP

(m,q)-CSP on alphabet Σ, n vars is defined by

m: # clauses

q: arity

$$x_1, x_2, \ldots, x_n \in \Sigma$$

$$f_1(x_{1,1}, x_{1,2}, \ldots x_{1,q}) = \text{TRUE}$$

$$f_2(x_{21}, x_{22}, \ldots, x_{2q}) = \text{TRUE}$$

$$\vdots$$

$$f_m(x_{m1}, x_{m2}, \ldots, x_{mq}) = \text{TRUE}$$

E.g. CHSH comes from a (4,2)-CSP over $\Sigma = \{0,1\}$, $n=4$ vars.

$$a_0, a_1, b_0, b_1$$

$$a_0 + b_0 = 0 \mod 2$$
$$a_0 + b_1 = 0 \mod 2$$
$$a_1 + b_0 = 0 \mod 2$$
$$a_1 + b_1 = 1 \mod 2$$

Given CSP Φ, $\exists$ MIP protocol G s.t.

$\omega(\Phi)$ = max frac. of satisfiable clauses

$\omega(G)$ = max acceptance prob.

1) $\omega(\Phi) = 1$

$\Rightarrow \omega(G) = 1$

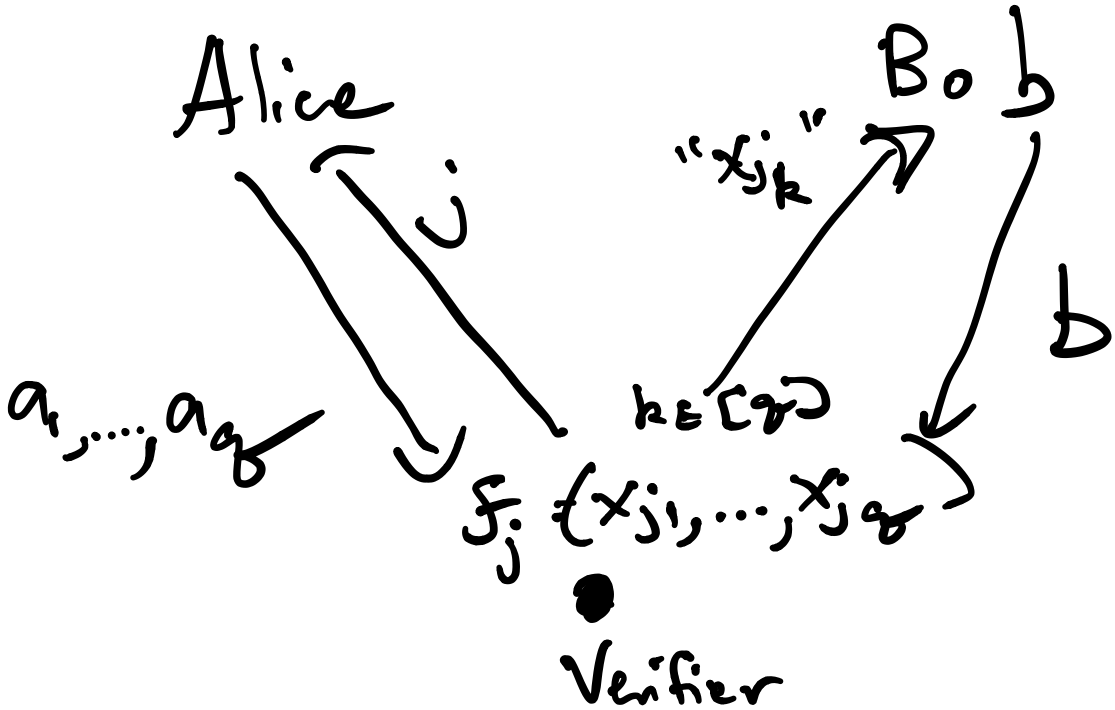
2) $\omega(\Phi) \leq 1/2$

$\Rightarrow \omega(G) \leq 1 - \frac{1}{10q^2}$

$\omega(\Phi) \le 1 - \varepsilon$.

$\Rightarrow \omega(G) \le 1 - \Omega(\varepsilon q^2)$

"Clause-variable game"



V accepts if

$$f_j(a_1, \ldots, a_q) = 1$$

$$a_k = b$$

- Magic square
- Linear systems games
- ...

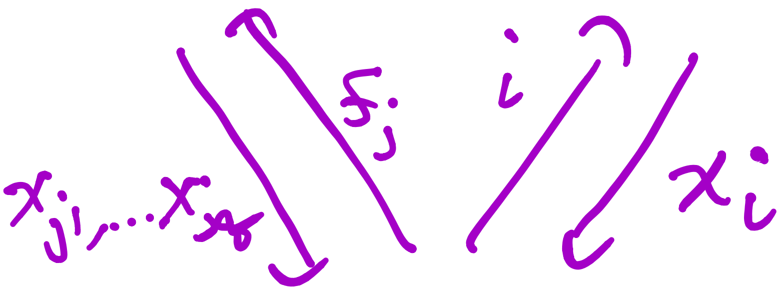
Pf:

$(\omega(\Phi)=1 \Rightarrow \omega(G)=1)$

1) $\exists\, x_1, \dots, x_n$

s.t. $f_1(x\dots) = f_2(\cdot) = \cdots$

$= f_m(x\dots) = \text{TRUE}$



V accepts if $f_j(x_{j1}\dots x_{jq}) = \text{TRUE}$ ✓

and $x_{jk} = x_i$ ✓

2) $\omega(\Phi) \le \underline{\underline{1/2}} \Rightarrow \omega(G) \le 1 - \frac{1}{\log^2}$ $= \sim$

Alice has a strategy

$j=1 \quad (a_{11}, a_{12}, \dots, a_{1q})$

$j=2 \quad (a_{21}, a_{22}, \dots, a_{2q})$

$\vdots$

Bob has a strategy

$b_1, \ldots, b_n$

$$\omega(G) > 1 - \frac{1}{10q^2} \Rightarrow \underbrace{\omega(\Phi)}_{} > \frac{1}{2}$$

Assignment to Φ will be Bob's strategy

$$(*) \quad \Pr_{j \in [m]}\left[f_j(b_{j_1}, \ldots, b_{j_q}) = \text{TRUE} \right]$$

$$\geq \Pr_j\left[(f_j[a_{j_1}, \ldots, a_{j_q}] = \text{TRUE}) \text{ AND } (a_{j_1} = b_{j_1}) \text{ AND} \ldots (a_{j_q} = b_{j_q}) \right]$$

$$\geq 1 - \Pr\Big[f_j[a_{j_1} \ldots a_{j_q}] = \text{FALSE} \Big]_{\leq \varepsilon}$$

$$- \Pr[a_{j_1} \neq b_{j_1}] - \ldots \Pr[a_{j_q} \neq b_{j_q}]$$

$\leq q \cdot \varepsilon$ $\leq q \cdot \varepsilon$

$\Pr[\text{A \& Bob win in } \mathcal{G}]$

$$= \Pr_{\substack{j \in [m] \\ k \in [q]}} \left[\left(f_j[a_{j_1} \dots a_{j_q}] = \text{TRUE} \right) \text{ AND } \left(a_{j_k} = b_{j_k} \right) \right]$$

$$> 1 - \varepsilon$$

$$\Rightarrow 1) \Pr_j \left[f_j[a_{j_1} \dots a_{j_q}] = \text{FALSE} \right] \leq \varepsilon$$

$$2) \Pr_{j, k \in [q]} \left[a_{j_k} \neq b_{j_k} \right] \leq \varepsilon$$

$$\Rightarrow \forall k, \quad \Pr_j \left[a_{j_k} \neq b_{j_k} \right] \leq q \cdot \varepsilon$$

$$(*) \geq 1 - \varepsilon - \underbrace{q \cdot \varepsilon - \dots - q \cdot \varepsilon}_{q \text{ times}}$$

$$= 1 - \varepsilon - q^2 \varepsilon = 1 - (q^2 + 1) \varepsilon$$

$$\varepsilon = \frac{1}{10 q^2} \Rightarrow (*) > 1/2 \quad \blacksquare$$