KIRBY CALCULUS

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1. Introduction to Kirby Calculus

The Lickorish-Wallace theorem states that any closed orientable 3-manifold can be obtained by an integral surgery on a link in S^3. The next natural question to ask is whether two links give rise to the same 3-manifold. It turns out that there is a nice classification of 3-manifolds first shown by Kirby [1] in 1978.

Theorem 1 (Kirby [1])**.** *Let L_1 and L_2 be two framed links, and let M_1 and M_2 be the 3-manifolds obtained by integral surgery, respectively. Then $M_1 \cong M_2$ if and only if L_2 can be obtained from L_1 by a sequence of Kirby moves.*

This is analogous to the fact that two knot diagrams are equivalent if one can be obtained from the other using a sequence of Reidemeister moves.

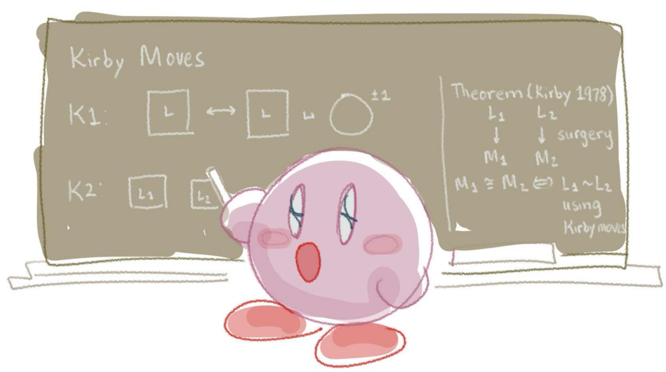

Figure 1. Kirby doing calculus. Courtesy of my friend daphgull.

Definition 1 (Kirby moves)**.** The *Kirby moves* $K1$ and $K2$ refer to the following two elementary operations on framed links.

Move *K*1. Add or delete an unknot with framing ± 1 that is disjoint from L.

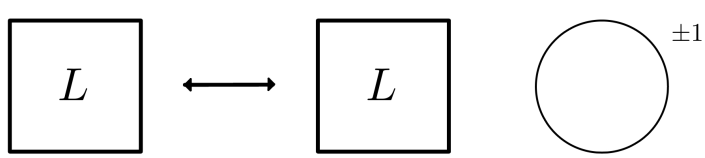

Figure 2. Kirby move $K1$.

Move $K2$. Let L_1 and L_2 be two link components of L with framing n_1 and n_2, respectively. Let L_2' be the framing n_2 of L_2.

Define $L_\# = L_1 \#_b L_2'$, where $\#_b$ refers to a band connecting L_1 and L_2', and the band is constructed so that the orientations of L_1 and L_2' agree.

Replace $L_1 \cup L_2$ by $L_\# \cup L_2$.

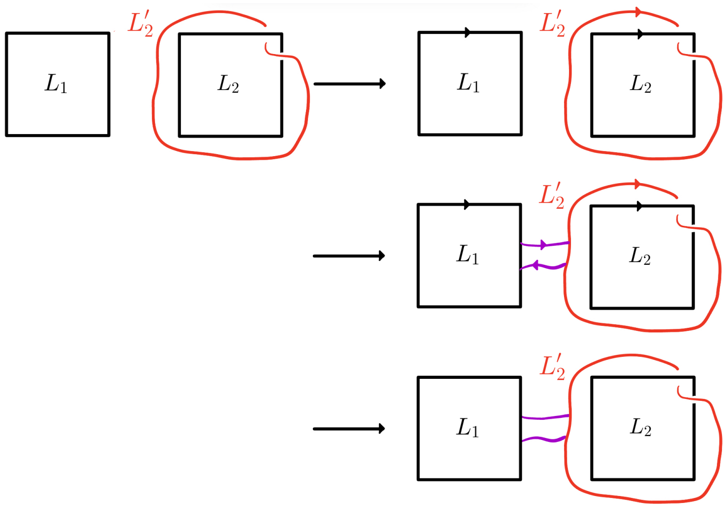

FIGURE 3. Kirby move $K2$.

It is clear that $K2$ changes L_1 but leaves L_2 unchanged. The next example shows how to construct the band and how L_1 changes depending on our chosen orientations.

Example 1. Consider the link 4_1^2 in Figure 4. Denote the left and right link components by L_1 and L_2 with framing numbers $n_1 = 3$ and $n_2 = 1$, respectively. Let L_1' and L_2' be the framings of L_1 and L_2, respectively.

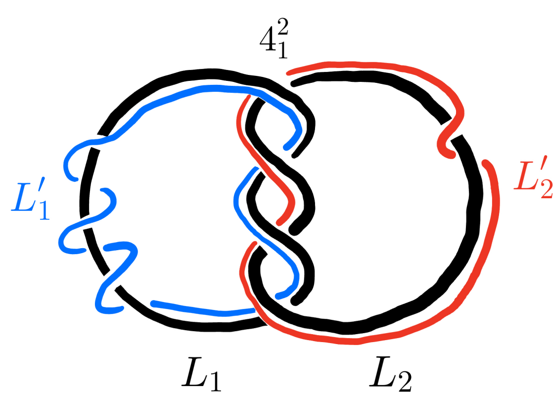

FIGURE 4. 4_1^2 with components L_1 and L_2 and framings L_1' and L_2'.

Now consider performing $K2$ on Figure 4. Without loss of generality, suppose we choose L_1 to be clockwise. There are two choices for the orientation of L_2.

Case 1 Suppose we choose the orientation of L_2 to be clockwise, shown in Figure 5.

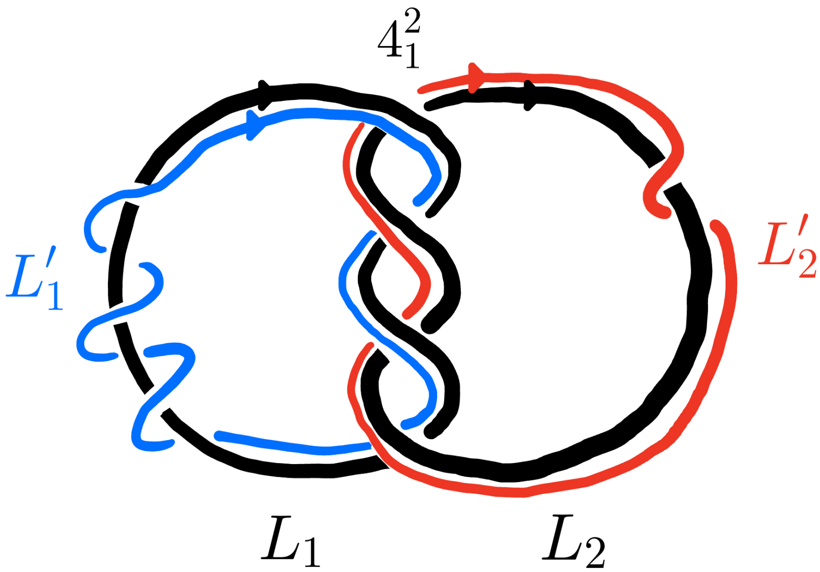

FIGURE 5. Clockwise orientation on L_2.

To construct the band sum $L_\# = L_1 \#_b L_2'$, we join L_1 and L_2' with bars such that the orientation is preserved. This band is shown in Figure 6.

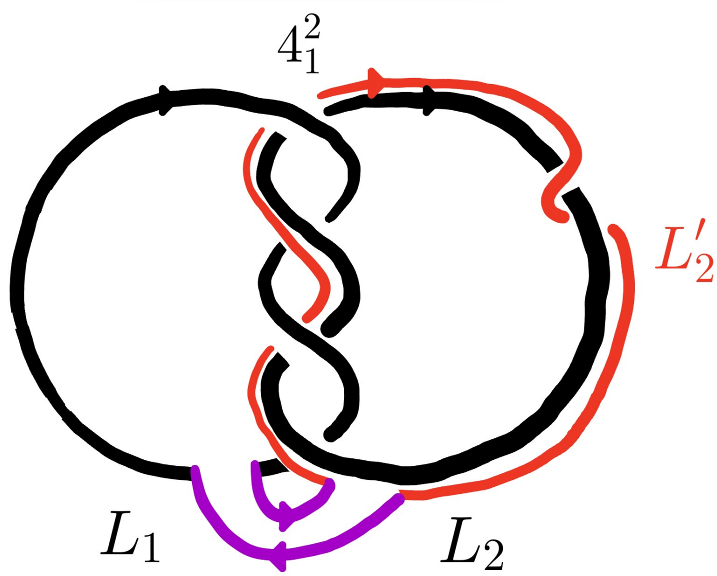

FIGURE 6. Constructing the $L_\# = L_1 \#_b L_2'$ (clockwise).

We remove L_1' for clarity. To finish the operation, we remove orientations. This results in $L_\#$ and L_2, shown in Figure 7.

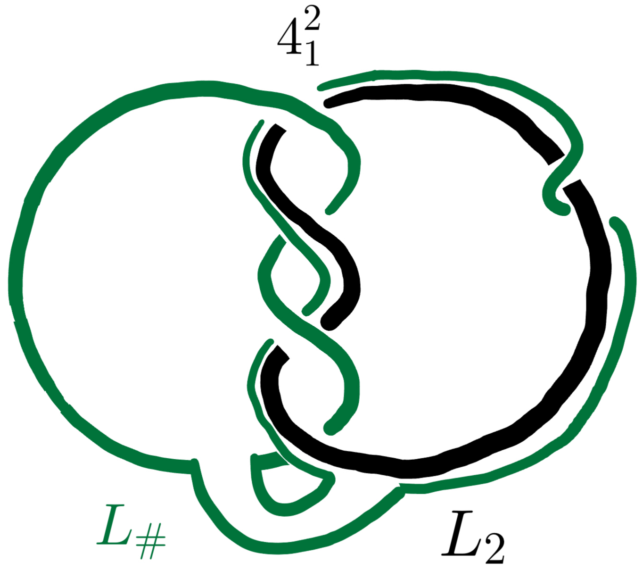

FIGURE 7. Applying $K2$ transforms L_1 into $L_\#$ (clockwise).

Case 2 Suppose we choose the orientation of L_2 to be counterclockwise, shown in Figure 8.

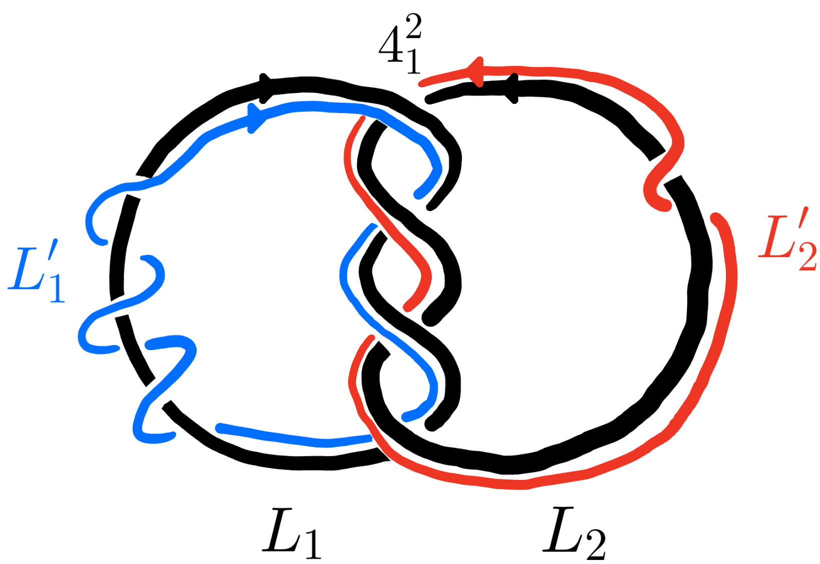

FIGURE 8. Counterclockwise orientation on L_2.

To construct the band sum $L_\# = L_1 \#_b L_2'$, we join L_1 and L_2' with bars such that the orientation is preserved. This band is shown in Figure 9. Note how this band is different from the one in Figure 6 due to our choice of orientation.

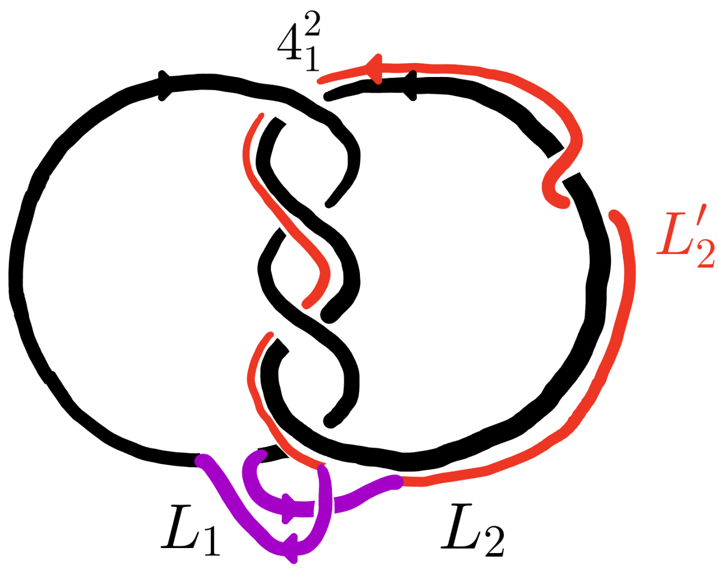

FIGURE 9. Constructing the $L_\# = L_1 \#_b L_2'$ (counterclockwise).

We remove L_1' for clarity. To finish the operation, we remove orientations. This results in $L_\#$ and L_2, shown in Figure 10.

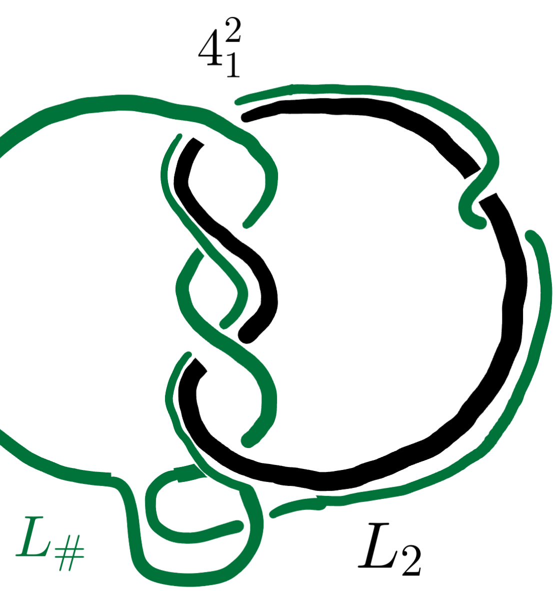

FIGURE 10. Applying $K2$ transforms L_1 into $L_\#$ (counterclockwise).

Example 1 corroborates the following.

Proposition 1. *Let L_1 and L_2 be link components framed by n_1 and n_2, respectively. Applying $K2$ results in*

$$n_1 \mapsto n_1 + n_2 + 2\,\mathrm{link}(L_1, L_2) \quad \textit{and} \quad n_2 \mapsto n_2. \tag{1}$$

Proof Sketch. As mentioned before, L_2 remains unchanged under $K2$, so the framing number n_2 is invariant. On the other hand, we replace L_1 with $L_\#$.

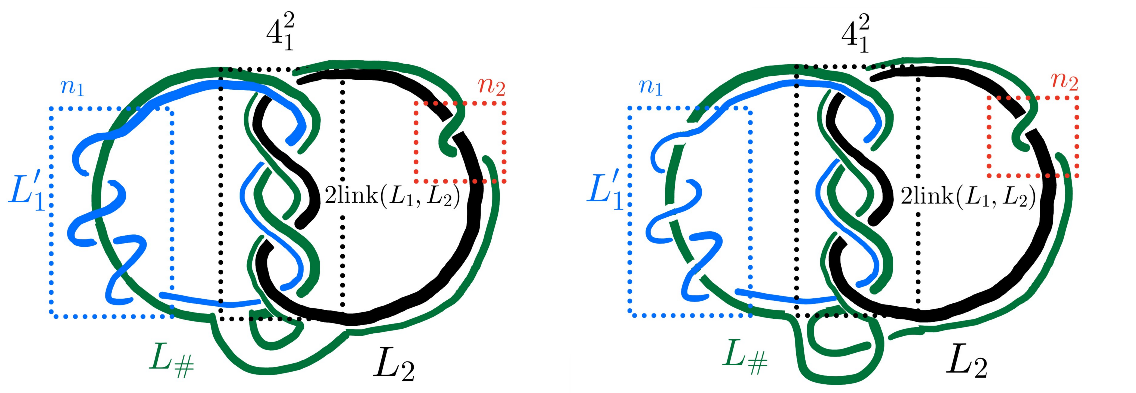

FIGURE 11. Contributions of each component to the framing of $L_\#$.

Recall that $L_\# = L_1 \#_b L_2'$. This gives information about the structure of the frame of $L_\#$.

- The frame of $L_\#$ contains the frame L_1'. This contributes framing number n_1. In the example, this is shown in the dotted blue box in Figure 11.
- To make $L_\#$ we band sum it with L_2'. Therefore, the frame of $L_\#$ inherently "twists" as many times as L_2', which is n_2. In the example, this is shown in the dotted red box in Figure 11.
- Finally, the band sum inherently links L_1 with L_2, and the number of "twists" is precisely $2\,\mathrm{link}(L_1, L_2)$. In the example, this is shown in the dotted black box in Figure 11.

Summing all the contributions gives the result. □

2. Key Results

Now we give a brief justification of Theorem 1. We only prove the ($\Longleftarrow$) direction; the ($\Longrightarrow$) direction is harder, and we refer the reader to [1] for the proof.

Proposition 2. *If L_2 can be obtained from L_1 through Kirby moves, the manifolds obtained from each link by integral surgery are homeomorphic.*

Proof. It suffices to show this claim for each Kirby move.

Move *K*1. Refer to Figure 2. Suppose we obtain M from L using integral surgery. We obtain M on the left-hand side and $M \# S^3$ on the right-hand side, and it is clear that $M \cong M \# S^3$.

Move $K2$. Refer to Figure 3. We have $L_1 \cup L_2 \mapsto L_\# \cup L_2$ under $K2$. Let M be the manifold obtained by integral surgery on L_2. Note M is obtained by gluing a torus in the exterior of L_2, where L_2' is glued to the boundary of a disc $D^2 \times \{\text{pt}\} \subseteq D^2 \times S^1$. Now L_1 and $L_\#$ are both disjoint from L_2, so they sit inside M. One can then isotope $L_\#$ into L_1 inside M by pushing it along $D^2 \times \{\text{pt}\}$.

The result follows. □

We end with a result that allows one to make major simplifications using Kirby moves.

Theorem 2. *An unknot with framing ± 1 can be removed from any link L.*

Proof. Repeatedly apply $K2$ to slide each arc over the unknot, and $K1$ removes it. □

Remark 1. One uses (1) to keep track of how the framing of each arc changes under Theorem 2.

To see Theorem 2 in action, we give examples with 1 and 2 arcs.

Example 2. Consider the link shown in Figure 12, where an arc of L_1 is passing through the unknot L_2 with frame L_2'.

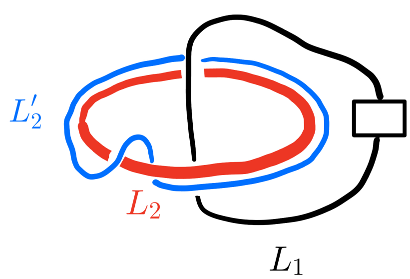

FIGURE 12. Arc passing through unknot of framing ± 1.

We assign orientations as shown in Figure 13

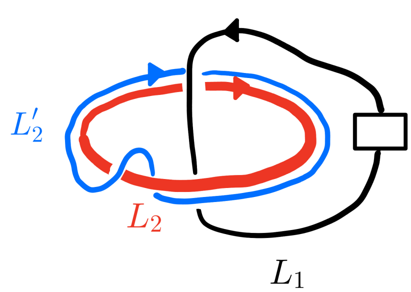

FIGURE 13. Assigning orientations to Figure 12.

We now apply a $K2$ move. The band sum connecting L_1 and L_2' is constructed so as to agree with the chosen orientations. This is shown in Figure 14.

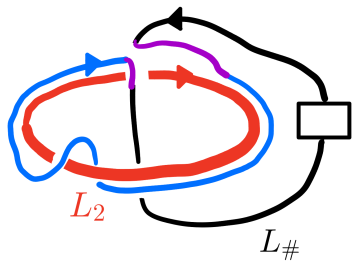

FIGURE 14. Applying a $K2$ move to Figure 13.

Now note $L_\#$ and L_2 are no longer linked, so we can separate them apart and remove L_2 using $K1$. See Figure 15.

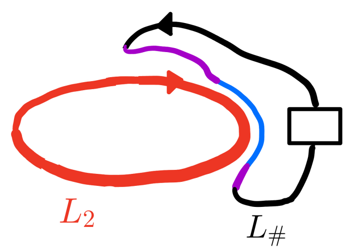

FIGURE 15. Separating $L_\#$ and L_2 from Figure 14.

Generally, by (1), an unknot with framing number -1 (resp. 1) increases (resp. decreases) the framing number of the arc by 1. This is shown in Figure 16.

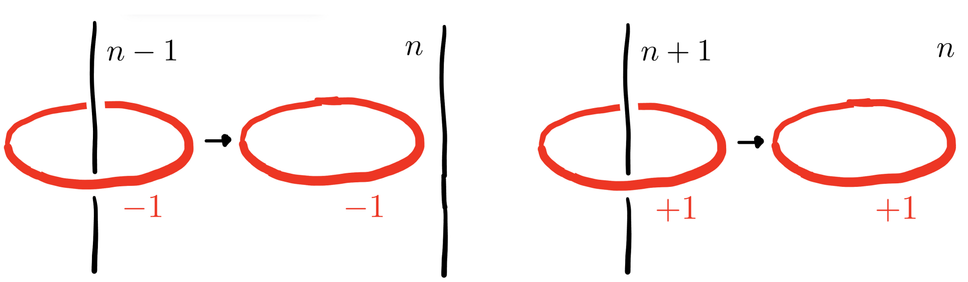

FIGURE 16. Change in framing numbers of one arc under unknots.

Example 3. Figure 17 shows that the two arcs twist around each other when removing the unknot using $K2$ moves.

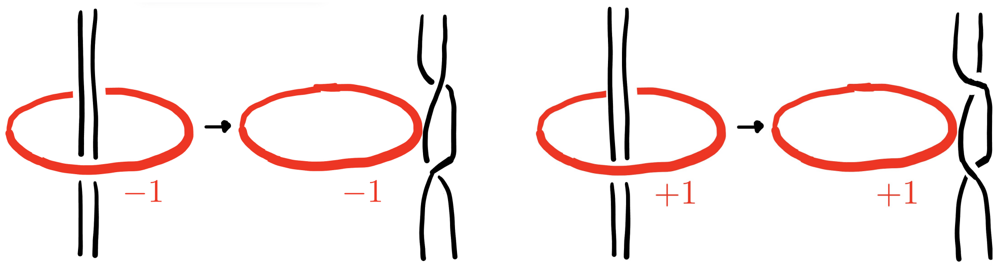

FIGURE 17. Twisting of two arcs under unknots.

For more examples, we refer to the reader to [2].

References

[1] R. Kirby, A calculus for framed links in S^3, *Invent. Math.* **45** (1978), 36–56.

[2] N. Saveliev, *Lectures on the Topology of* 3*-Manifolds: An Introduction to the Casson Invariant*, Walter de Gruyter GmbH & Co. KG, 10785 Berlin/Boston, 2012.

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