CHAPTER V: INTEGRAL QUADRATIC FORMS WITH DISCRIMINANT ± 1
§2.3: THE DEFINITE CASE

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1. Introduction

In the indefinite case, two forms are equivalent if and only if they have have the same rank, index, and type. A natural question to ask is if there is a classification for definite forms. It turns out this question is hard to answer. However, we do know that there is a finite number of definite classes for each rank.

Theorem 1 (Finiteness Theorem)**.** *For each n, S_n has finitely many definite classes.*

Over the past century, the efforts of many mathematicians led to the determination of these definite classes for small $n \in \mathbb{N}$. For example, Kneser [2] classified the definite forms for ranks $n \leq 16$ in 1957.

Theorem 2 (Kneser 1957, [2])**.** *The number of rank n definite classes for $n \leq 16$ are*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
#	1	1	1	1	1	1	1	2	2	2	2	3	3	4	5	8

In the sequel we focus on definite quadratic forms of type II. Note definite quadratic forms of type II are necessarily rank 8 (see Serre [7] §5.2.1 Corollary 2 p. 53). Let C_n be the set of isomorphism classes of the elements of S_n that are positive definite and type II, and let G_E be the automorphism group of $E \in S_n$. Define the Eisenstein mass by

$$M_n := \sum_{E \in C_n} \frac{1}{|G_E|}.$$

The Eisenstein mass has a useful closed form when $8 \mid n$.

Proposition 1 (Minkowski-Siegel formula)**.** *Let B_j be the jth Bernoulli number.*[1] *Let $n = 8k$ for some $k \in \mathbb{N}$. Then*

$$M_n = \frac{B_{2k}}{8k} \prod_{j=1}^{4k-1} \frac{B_j}{4j}.$$

[1]Conventionally the Bernoulli numbers are coefficients of the generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Determining the coefficients from the generating function gives, for example, $B_1 = 1/2$, $B_2 = 1/6$, $B_4 = 1/30$, $B_6 = 1/42$, and $B_j = 0$ if j is an odd integer greater than 1. Serre, however, indexes only the even indices; hence Serre's convention gives $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, and so on.

2. Rank 8 Forms

The Minkowski-Siegel formula is useful in classifying equivalent forms; specifically, it gives a method to prove that a subset of the classes in C_n is precisely C_n. For the case $k = 1$ $(n = 8)$ we have the following.

Theorem 3 (Mordell 1938, [6]). *Every rank* 8 *definite form of type II is equivalent to* Γ_8.

Proof. The Minkowski-Siegel formula gives

$$\sum_{E \in C_8} \frac{1}{|G_E|} = M_8 = \frac{B_2}{8} \prod_{j=1}^{3} \frac{B_j}{4j} = \frac{1/6}{8} \cdot \frac{1/30}{4} \cdot \frac{1/42}{8} \cdot \frac{1/30}{12} = \frac{1}{696729600}.$$

Note Γ_8 is a representative in C_8, and $|G_{\Gamma_8}| = 696729600$. Therefore, Γ_8 is the only representative in C_8, which gives the result. □

3. Rank 16 Forms

For the case $k = 2$ $(n = 16)$, we have the following.

Theorem 4 (Witt 1941, [8]). *Every rank* 16 *definite form of type II is equivalent to etiher* Γ_{16} *or* $\Gamma_8 \oplus \Gamma_8$.

Proof. As before, the Minkowski-Siegel formula gives

$$\sum_{E \in C_{16}} \frac{1}{|G_E|} = M_{16} = \frac{B_4}{16} \prod_{i=1}^{7} \frac{B_j}{4j} = \frac{1}{2^{15} \cdot 16!} + \frac{1}{2^{29} \cdot 3^{10} \cdot 5^4 \cdot 7^2}.$$

Both Γ_{16} and $\Gamma_8 \oplus \Gamma_8$ are representatives in C_{16} with respective orders $|G_{\Gamma_{16}}| = 2^{15} \cdot 16!$ and $|G_{\Gamma_8 \oplus \Gamma_8}| = 2^{29} \cdot 3^{10} \cdot 5^4 \cdot 7^2$. Therefore, Γ_{16} and $\Gamma_8 \oplus \Gamma_8$ are the only representatives in C_{16}, which gives the result. □

4. Rank 24 Forms

There are many more cases to consider when $k = 3$ $(n = 24)$. The same approach gives the following.

Theorem 5 (Niemeier 1973, [3]). *There are* 24 *rank* 24 *definite forms of type II.*

In particular, the Leech lattice Λ_{24} is one of the representatives for the set of equivalent rank 24 equivalent forms. Leech [5] used Λ_{24} for sphere packing in $\mathbb{R}^{24}$. Interestingly, Λ_{24} contains no vector x such that $x \cdot x = 2$. Moreover, the quotient of the Leech lattice by its center gives Co_1, one of the 26 sporadic simple groups.

Theorem 6 (Conway 1969, [1]). *The group* $\mathrm{Co}_1 := G_{\Lambda_{24}}/\{\pm 1\}$ *is a sporadic simple group of order* $|\mathrm{Co}_1| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

5. Rank 32 Forms and Beyond

The number of definite forms of type II increases rapidly as $n \geq 25$. The Minkowski-Siegel formula gives a lower bound of the mass M_{32}

$$M_{32} \approx 4.03092 \cdot 10^7 > 4 \cdot 10^7.$$

For each E, we have $|G_E| > 2$, so $1/|G_E| < 1/2$. Hence each form E contributes less than $1/2$ to the mass M_{32}, so the class C_{32} has more than $8 \cdot 10^7$ elements. They have not been classified yet. Recently in 2003, King showed that there are significantly more elements in C_{32} than the simple bound calculated from Minkowski-Siegel.

Theorem 7 (King 2003, [4])**.** *The number of elements* $|C_{32}| > 1 \cdot 10^9$.

The table below shows the approximate value of the masses, which shows the rate at which the number of definite forms of type II increases as the rank increases.

n	$\approx M_n$
8	$1.43528 \cdot 10^{-9}$
16	$2.48859 \cdot 10^{-18}$
24	$7.93678 \cdot 10^{-15}$
32	$4.03092 \cdot 10^{7}$
40	$4.39308 \cdot 10^{51}$
48	$1.52575 \cdot 10^{121}$
56	$6.38119 \cdot 10^{218}$
64	$4.77227 \cdot 10^{346}$
72	$4.80657 \cdot 10^{506}$
80	$2.93101 \cdot 10^{700}$
88	$3.24266 \cdot 10^{929}$
96	$1.40595 \cdot 10^{1195}$

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