CHAPTER IV: QUADRATIC FORMS OVER $\mathbb{Q}_p$
§2.3: CLASSIFICATION

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1. Preliminaries

We have seen many properties of quadratic forms. In particular, we have seen two invariants of quadratic forms.

Proposition 1. *Let Q be a quadratic form over a field k. The discriminant of the quadratic form* $\operatorname{disc}(Q) \in k^*/(k^*)^2$ *is independent of our choice of basis.*

Proposition 2. *Let $e = (e_1, \ldots, e_n)$ be an orthogonal basis for a quadratic form Q over a field k, and let $a_i = e_i \cdot e_i$. Then the* Hasse-Witt invariant

$$\varepsilon(Q) = \varepsilon(e) := \prod_{i<j} (a_i, a_j)$$

is independent of our choice of orthogonal basis.

Finally, we recall a fact stating when a quadratic form Q over a field k represents an element $a \in k^*$.

Proposition 3. *Let $a \in k^*$. A quadratic form Q over a field k represents a if and only if one of the following is true:*

i) $\operatorname{rank}(Q) = 1$ *and* $a = \operatorname{disc}(Q)$*;*
ii) $\operatorname{rank}(Q) = 2$ *and* $(a, -\operatorname{disc}(Q)) = \varepsilon(Q)$*;*
iii) $\operatorname{rank}(Q) = 3$ *and* $a \neq -\operatorname{disc}(Q)$*;*
iv) $\operatorname{rank}(Q) = 3$*,* $a = -\operatorname{disc}(Q)$*, and* $(-1, -\operatorname{disc}(Q)) = \varepsilon(Q)$*;*
v) $\operatorname{rank}(Q) \geq 4$.

(Note that all the identities are in $k^*/(k^*)^2$.)

Proposition 3 then implies that if there exists another quadratic form Q' over k such that $\operatorname{disc}(Q) = \operatorname{disc}(Q')$ and $\varepsilon(Q) = \varepsilon(Q')$, then Q' also represents a. This leads to the following corollary.

Corollary 1. *If Q and Q' have the same invariants, then they represent the same elements.*

With these useful invariants and properties at hand, we are now ready to classify quadratic forms over k up to equivalence.

2. Classification

2.1. **Equivalence.**

Theorem 1. *Two quadratic forms Q and Q' over k are equivalent if and only if* $\operatorname{rank}(Q) = \operatorname{rank}(Q')$, $\operatorname{disc}(Q) = \operatorname{disc}(Q')$, *and* $\varepsilon(Q) = \varepsilon(Q')$.

Proof. The ($\Longrightarrow$) direction is straightforward, but we give it here for the sake of completeness. Suppose Q and Q' are equivalent over k. Let $\mathscr{B}$ and $\mathscr{B}'$ be corresponding orthogonal bases for Q and Q', and let A and A' be the matrices of Q and Q' with respect to $\mathscr{B}$ and $\mathscr{B}'$, respectively. Because $Q \sim Q'$, there exists an invertible matrix X (that changes $\mathscr{B}$ to $\mathscr{B}'$) such that

$$A' = XAX^t. \tag{1}$$

From (1), we obtain $\operatorname{rank}(Q) = \operatorname{rank}(Q')$; Propositions 1 and 2 give $\operatorname{disc}(Q) = \operatorname{disc}(Q')$ and $\varepsilon(Q) = \varepsilon(Q')$ since our invariants are independent of choice of orthogonal basis.

We prove the ($\Longleftarrow$) direction by induction on $n = \operatorname{rank}(Q) = \operatorname{rank}(Q')$. If $n = 0$, then there is nothing to show. Suppose our statement is true for quadratic forms of rank $n-1$. Let Q and Q' have rank n, $\operatorname{disc}(Q) = \operatorname{disc}(Q')$, and $\varepsilon(Q) = \varepsilon(Q')$. Because the invariants are the same, corollary 1 implies that Q and Q' represents some $a \in k^*$. Hence we write

$$Q \sim aX^2 \oplus R \quad \text{and} \quad Q' \sim aX^2 \oplus R',$$

where $\operatorname{rank}(R) = \operatorname{rank}(R') = n - 1$. It suffices to show that $R \sim R'$ since that would imply $Q \sim Q'$. To this end, note that the invariants of R and R' are the same,[1] as

$$\operatorname{disc}(R) = a \operatorname{disc}(Q) = a \operatorname{disc}(Q') = \operatorname{disc}(R') \tag{2}$$

and

$$\varepsilon(R) = \varepsilon(Q)(a, \operatorname{disc}(R)) = \varepsilon(Q')(a, \operatorname{disc}(R')) = \varepsilon(R'). \tag{3}$$

Therefore, our induction hypothesis implies $R \sim R'$, as desired. □

2.2. **Existence.**

Theorem 2. *Let $n \in \mathbb{N}$, $d \in k^*/(k^*)^2$, and $\varepsilon = \pm 1$. There exists a quadratic form Q over k of rank n such that* $\operatorname{disc}(Q) = d$ *and* $\varepsilon(Q) = \varepsilon$ *if and only if one of the following is true:*

i) $n = 1$, $\varepsilon = 1$;
ii) $n = 2$, $d \neq -1$;
iii) $n = 2$, $\varepsilon = 1$;
iv) $n \geq 3$.

Proof. We proceed with casework on the rank n. In the sequel, when we say that a quadratic form Q satisfies the request, we mean that $\operatorname{disc}(Q) = d$ and $\varepsilon(Q) = \varepsilon$ for the given choice of d and ε.

[1] In (2), $\operatorname{disc}(R) = a \operatorname{disc}(Q)$ because squares vanish in $k^*/(k^*)^2$ and $\operatorname{disc}(Q)$ has a factor of a. Similarly, in (3), recall that the Hilbert symbol $(\cdot,\cdot)$ defines a map $k^*/(k^*) \times k^*/(k^*) \to \{\pm 1\}$, so squares inside the Hilbert symbols vanish.

Case 1 Consider $n = 1$. Then $Q \sim aX^2$ for some $a \in k^*/(k^*)^2$, and $\varepsilon(Q) = 1$ since the Hasse-Witt invariant is the empty product, so i) holds. Conversely, if $\varepsilon = 1$, then $Q = dX^2$ satisfies the request.

Case 2 Consider $n = 2$. Then $Q \sim aX^2 + bY^2$ for some $a, b \in k^*/(k^*)^2$. Hence $\operatorname{disc}(Q) = ab$ and $\varepsilon(Q) = (a, b)$. Note that if $\operatorname{disc}(Q) = -1$, then

$$\varepsilon(Q) = (a, b) = (a, -ab) = (a, 1) = 1, \tag{4}$$

which means that we cannot have $\operatorname{disc}(Q) = -1$ and $\varepsilon(Q) = -1$ simultaneously for any form Q of rank 2. Thus, if $\operatorname{disc}(Q) \neq -1$, then ii) holds; otherwise, $\varepsilon = 1$, and iii) holds.

Conversely, if $d \neq -1$, then we choose some $a \in k^*$ such that $(a, -d) = \varepsilon$; $Q = aX^2 + adY^2$ satisfies the request. If $d = -1$ and $\varepsilon = 1$, then $Q = X^2 - Y^2$ satisfies the request.

Case 3 Consider $n = 3$. Then any quadratic form of rank n satisfies iv). Conversely, choose some $a \in k^*/(k^*)^2$ distinct from $-d$. There exists a quadratic form R of rank 2 such that $\operatorname{disc}(R) = ad \neq -1$ and $\varepsilon(R) = \varepsilon(a, -d)$, so $Q = aX^2 \oplus R$ satisfies the request.

Case 4 Consider $n \geq 4$. Then any quadratic form of rank n satisfies iv). Conversely, choose some quadratic form R of rank 3 that satisfies the request. Then the quadratic form $Q = R(X_1, X_2, X_3) + X_4^2 + \cdots + X_n^2$ has rank n and also satisfies the request since the quadratic form $X_4^2 + \cdots + X_n^2$ has discriminant 1 and Hasse-Witt invariant 1.

Having exhausted all cases, we are done. □

We can now use Theorems 1 and 2 to classify the quadratic forms over $k = \mathbb{Q}_p$.

Corollary 2. *The number of quadratic forms over $\mathbb{Q}_p$ of rank n up to equivalence is shown below.*

n	$\mathbb{Q}_2$	$\mathbb{Q}_p,\ p > 2$
1	8	4
2	15	7
≥ 3	16	8

Proof. Recall that $\left|\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2\right| = 8$ and $\left|\mathbb{Q}_p^*/\left(\mathbb{Q}_p^*\right)^2\right| = 4$ for $p > 2$.[2] We analyze $\mathbb{Q}_2$.

Case 1 Theorem 2 asserts that a form Q over $\mathbb{Q}_2$ of rank 1 has $(\operatorname{disc}(Q), \varepsilon(Q)) = (d, 1)$ for any $d \in \mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$, and there are $8 \cdot 1 = 8$ such pairs.

Case 2 Theorem 2 asserts that a form Q over $\mathbb{Q}_2$ of rank 2 has $(\operatorname{disc}(Q), \varepsilon(Q)) = (d, \varepsilon)$ for any $d \in \mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ and any $\varepsilon = \pm 1$, but $(\operatorname{disc}(Q), \varepsilon(Q)) \neq (-1, -1)$. Thus, there are $8 \cdot 2 - 1 = 16 - 1 = 15$ such pairs.

Case 3 Theorem 2 asserts that a form Q over $\mathbb{Q}_2$ of rank 3 has $(\operatorname{disc}(Q), \varepsilon(Q)) = (d, \varepsilon)$ for any $d \in \mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ and any $\varepsilon = \pm 1$, and there are $8 \cdot 2 = 16$ such pairs.

The $\mathbb{Q}_p$ case for when $p > 2$ is similar, except that d can only take on a maximum of 4 values. □

[2] Note that $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ and $\mathbb{Q}_p^*/\left(\mathbb{Q}_p^*\right)^2$ are three- and two-dimensional vector spaces over $\mathbb{F}_2$.

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