Values of Dirichlet Series

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Abstract

We study several classes of Dirichlet series, some of which were studied by Shimura, Choi, and others. We prove the meromorphic continuation of the series. We also study the values of the series at certain integers. We generalize results of Choi by relating values of some series to values of other, related series.

1 Introduction

Number theory is one of the oldest branches of mathematics. It is devoted to the study of integers and integer-valued functions. The main area of study is prime numbers. An integer $n > 1$ is prime if n is divisible only by 1 and itself. If $n > 1$ and n is not prime, then n is composite. These prime numbers are significant because every integer $n > 1$ can be represented uniquely as a product of primes, up to the order of factors. This is known as the Fundamental Theorem of Arithmetic, first noted by Euclid in his *Elements* [11]. Because any natural number can be factored into prime numbers, prime numbers have significant contributions in cryptography. One useful application of the factorization of numbers into primes is the Rivest–Shamir–Adleman (RSA) encryption system, which involves finding the prime factorization of large numbers. There are no trivial algorithms that can perform this efficiently when the numbers are sufficiently large, making RSA useful for electronic transactions involving sensitive information such as credit card numbers and bank account information.

Because prime numbers are fundamental, it is of great consequence to study their properties. A well-known result by Euclid in his *Elements* states that there is an infinite number of primes [11]. Other properties of prime numbers are often derived by studying the Riemann zeta function $\zeta(s)$ and Dirichlet L-functions. Euler, in fact, studied the behavior of $\zeta(s)$ using calculus. In his 1737 paper, he showed that the Riemann zeta function can be represented as an infinite product over the primes [7]. This infinite product is known as the

Euler product. Let $\Re(s)$ denote the real part of $s \in \mathbb{C}$. Then for $\Re(s) > 1$,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Therefore, $\zeta(s)$ has rich information about primes. In fact, a simple pole of $\zeta(s)$ at $s = 1$ implies Euclid's infinitude of primes.

Studying $\zeta(s)$ includes finding its special values, a functional equation, and a meromorphic continuation. Another well-known result is the *Basel problem*, which prompts the exact sum of $\zeta(2)$. Euler [6] found that

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$$

and the value of the Riemann zeta function at every positive even integer; specifically, for every $n \in \mathbb{N}$, where B_n are the Bernoulli numbers,

$$\zeta(2n) = \frac{(-1)^{n-1} 2^{n-1} B_{2n} \pi^{2n}}{(2n)!}.$$

In his memoir in 1859, Riemann [14] expanded the definition of the Riemann zeta function and found its meromorphic continuation and functional equation. The meromorphic continuation of $\zeta(s)$ is necessary for the form of the functional equation such that the extension exists everywhere [1]. Specifically, he found

$$\xi(s) = \xi(1-s) \quad \text{where} \quad \xi(s) = \frac{1}{2} \pi^{-s/2} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

It is due to the study of these properties of $\zeta(s)$ that provided the analytical tools needed for Hadamard [8] and de la Vallée Poussin [4] to prove the classical Prime Number Theorem on the distribution of primes. Specifically, if $\pi(x)$ denotes the number of primes less than or equal to x, then

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1.$$

The Riemann Hypothesis is considered to be the most important unsolved problem in mathematics. It conjectures that the real part of every zero of $\zeta(s)$ is $1/2$, notwithstanding the trivial zeroes at every negative even integer. The line $\Re(s) = 1/2$ is known as the critical line. The consequence of the Riemann Hypothesis is enormous in the distribution of prime numbers, as it not only implies the best possible bound of $\pi(x)$, due to Von Koch [18], but also the growth of other arithmetical functions, among others. Hardy [10] and Hardy & Littlewood [9] proved that there are infinitely many zeroes on the critical line.

Dirichlet L-functions are similar objects to the Riemann zeta function. A Dirichlet L-function is a function of the form

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Similar to the Riemann zeta function, $L(s,\chi)$ has rich information about primes. In fact, Dirichlet used a property of $L(s,\chi)$ to prove his celebrated theorem on primes in arithmetic progressions [5] that there are infinitely many primes ending in any one of the digits 1, 3, 7, or 9.

We turn our study to the *Choi Dirichlet series* and the *Choi Alternating Dirichlet Series*, which were studied by Shimura [15]. For $s \in \mathbb{C}$ with $\Re(s) \geq 1$, $r \in \{0,1\}$, and $a,b \in \mathbb{N}$, we define the non-alternating Choi Dirichlet series

$$D(s,a,b) \coloneqq \sum_{n=0}^{\infty} \left(\frac{1}{(an+b)^s} + \frac{(-1)^s}{(an+a-b)^s} \right) \tag{1}$$

and the alternating Choi Dirichlet series with $\Re(s) \geq 1$

$$D_{\text{Choi}}(s,a,b) \coloneqq \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(an+b)^s} + \frac{(-1)^{s-1}}{(an+a-b)^s} \right) \tag{2}$$

for $0 \leq b \leq a/2$. In particular, we obtain formulas for special values of those Dirichlet series and their meromorphic continuations. Choi [3] obtained special values for $D_{\text{Choi}}(2,a,b)$ for

$$(a,b) \in \{(4,1),(8,1),(12,1),(16,1),(20,1)\}.$$

These values are

$$\begin{aligned} D_{\text{Choi}}(2,4,1) &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{(4n+1)^2} - \frac{1}{(4n+3)^2} \right) \\ &= \frac{\pi^2}{8\sqrt{2}}, \end{aligned}$$

$$\begin{aligned} D_{\text{Choi}}(2,8,1) &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{(8n+1)^2} - \frac{1}{(8n+7)^2} \right) \\ &= \frac{1}{32}\sqrt{5+\frac{7}{\sqrt{2}}}\pi^2, \end{aligned}$$

$$\begin{aligned} D_{\text{Choi}}(2,12,1) &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{(12n+1)^2} - \frac{1}{(12n+11)^2} \right) \\ &= \frac{1}{72}\sqrt{26+15\sqrt{3}}\pi^2, \end{aligned}$$

$$\begin{aligned} D_{\text{Choi}}(2,16,1) &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{(16n+1)^2} - \frac{1}{(16n+15)^2} \right) \\ &= \frac{1}{128}\sqrt{42+29\sqrt{2}+\sqrt{3445+\frac{4871}{\sqrt{2}}}}\pi^2, \text{ and} \end{aligned}$$

$$
\begin{aligned}
D_{\text{Choi}}(2,20,1) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(20n+1)^2}-\frac{1}{(20n+19)^2}\right)\\
&= \frac{\pi^2}{400}\cot\left(\frac{\pi}{20}\right)\csc\left(\frac{\pi}{20}\right).
\end{aligned}
$$

As a consequence of formulas from [2], we obtain the following.

Theorem 1. *Let $a, b \in \mathbb{N}$. For $n \in \mathbb{N}$,*

$$
D_{\text{Choi}}(n,a,b) = \frac{\pi^n}{a^n(n-1)!}\csc^n\left(\frac{b\pi}{a}\right)P_{n-1}\left(\cos\left(\frac{b\pi}{a}\right)\right), \tag{3}
$$

where

$$
P_n(x) = \sum_{k=0}^{\lfloor\frac{n+2}{2}\rfloor} T(n,k)x^{n-2k}.
$$

As examples,

$$
D_{\text{Choi}}(5,3,2) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(3n+2)^5}+\frac{1}{(3n+1)^5}\right) = \frac{17\pi^5}{2916\sqrt{3}}
$$

and

$$
D_{\text{Choi}}(6,4,1) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(4n+1)^6}-\frac{1}{(4n+3)^6}\right) = \frac{361\pi^6}{245760\sqrt{2}}.
$$

In Theorem 1, $T(n,k)$ refers to the number of permutations of $\{1,2,\ldots,n\}$ with k increasing runs of length at least 2. Recursively,

$$
\begin{gathered}
T(n,0)=1, \qquad T(n,k)=0 \quad \left(k>\left\lfloor\frac{n}{2}\right\rfloor\right),\\
T(n+1,k)=(2k+1)T(n,k)+(n-2k+2)T(n,k-1).
\end{gathered} \tag{4}
$$

As an example, $T(3,1)=5$ because there are 6 permutations of $\{1,2,3\}$, but only $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, and $\{3,1,2\}$ have 1 subset of length 2 that is increasing. The final permutation $\{3,2,1\}$ is not increasing. Moreover, $T(2n,n)$ gives the Euler numbers. For more information, see [17].

Using similar methods, we also obtain the following theorem.

Theorem 2. *Let $a, b \in \mathbb{N}$. For $n \in \mathbb{N}$,*

$$
D(n,a,b) = \frac{\pi^n}{a^n(n-1)!}\csc^n\left(\frac{b\pi}{a}\right)Q_{n-1}\left(\cos\left(\frac{b\pi}{a}\right)\right), \tag{5}
$$

where

$$
Q_n(x) = \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor} t(n,k)x^{n-2k+1} \quad \textit{for} \quad n\geq 1 \quad \textit{and} \quad Q_0(x)=x.
$$

In Theorem 2, $t(n,k)$ refers to the number of permutations of $\{1,\ldots,n-1\}$ with k peaks. For example, $t(4,1)=4$ because there are 6 permutations of $\{1,2,3\}$, but only $\{1,2,3\}$, $\{1,3,2\}$, $\{2,3,1\}$, and $\{3,2,1\}$ have 1 peak. The other two, $\{2,1,3\}$ and $\{3,1,2\}$, have 2 peaks. Recursively,

$$\begin{aligned} &t(n,1)=2^{n-1}, \qquad t(1,k)=0 \quad (k>1),\\ &t(n,k)=(n-2k+2)t(n-1,k-1)+2kt(n-1,k). \end{aligned} \tag{6}$$

For more information, see [16]. Finally, we have the meromorphic continuation of $D_{\text{Choi}}(s,a,b)$.

Theorem 3. *$D_{\text{Choi}}(s,a,b)$ has a meromorphic continuation.*

A proof of this is given in the next section.

2 Meromorphic Continuation of Choi Dirichlet Alternating Series

We prove Theorem 3. First, we provide the definition of the Lerch zeta function [13], essentially one of the many generalizations of the Riemann zeta function. The Lerch zeta function is denoted by $L(\lambda,\alpha,s)$ for $\Re(\alpha)>0$ and $\lambda\in\mathbb{R}$ and is given by

$$L(\lambda,\alpha,s)=\sum_{n=0}^{\infty}\frac{e^{2\pi i n\lambda}}{(n+\alpha)^s}.$$

Like $\zeta(s)$, the Lerch zeta function has a meromorphic continuation. Now we are ready to prove Theorem 3.

Proposition 1. *Let $a,b\in\mathbb{N}$ and let $s\in\mathbb{C}$ with $\Re(s)\geq 1$. Then $D_{\text{Choi}}(s,a,b)$ is expressible as the sum of two Lerch zeta functions.*

Proof. We write

$$\begin{aligned} D_{\text{Choi}}(s,a,b) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(an+b)^s}+\frac{(-1)^{s-1}}{(an+a-b)^s}\right)\\ &= \frac{1}{a^s}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{b}{a}\right)^s}+\sum_{n=0}^{\infty}\frac{(-1)^{s-1+n}}{\left(n+\frac{a-b}{a}\right)^s}\right)\\ &= \frac{1}{a^s}\left(\sum_{n=0}^{\infty}\frac{e^{2\pi i\left(\frac{1}{2}\right)n}}{\left(n+\frac{b}{a}\right)^s}+\sum_{n=0}^{\infty}\frac{e^{2\pi i(s-1+n)\left(\frac{1}{2}\right)}}{\left(n+\frac{a-b}{a}\right)^s}\right)\\ &= \frac{1}{a^s}\left(L\left(s,\frac{1}{2},\frac{b}{a}\right)+L\left(s,\frac{s-1+n}{2},\frac{a-b}{a}\right)\right), \end{aligned}$$

as required. ■

Proposition 1, in essence, is sufficient to prove Theorem 3 because now $D_{\text{Choi}}(s,a,b)$ has a meromorphic continuation in the form of the mermomorphic continuations of two Lerch zeta functions.

3 Special Values of Choi Dirichlet Alternating Series

Here, we prove Theorem 1, the general formula that calculates special values of $D_{\text{Choi}}(s,a,b)$ on positive integers. The proof ultimately relies on the following two propositions.

Proposition 2. *Let $a,b \in \mathbb{N}$. Then*

$$(D_{\text{Choi}}(1,a,b))^2 = D(2,a,b). \tag{7}$$

Proof. Let $z=b/a$. Then (7) is equivalent to

$$\left(D_{\text{Choi}}\left(1,1,z\right)\right)^2 = D\left(2,1,z\right). \tag{8}$$

For the sake of brevity, let

$$A_N = D_{\text{Choi}}\left(1,1,z\right) = \sum_{n=0}^{N} (-1)^n \left(\frac{1}{n+z}\right)$$

and

$$B_N = D\left(2,1,z\right) = \sum_{n=0}^{N} \frac{1}{\left(n+z\right)^2}.$$

As

$$\begin{aligned} A_N^2 = \left(\sum_{n=0}^{N} (-1)^n \left(\frac{1}{n+z}\right)\right)^2 &= \sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{\left(n+z\right)\left(m+z\right)} + \sum_{n=0}^{N} \frac{1}{\left(n+z\right)^2} \\ &= \sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{\left(n+z\right)\left(m+z\right)} + B_N, \end{aligned}$$

we therefore have

$$\begin{aligned} A_N^2 - B_N &= \sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{\left(n+z\right)\left(m+z\right)} = \sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{m-n} \left(\frac{1}{n+z} - \frac{1}{m+z}\right) \\ &= \sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{\left(m-n\right)\left(n+z\right)} - \sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{\left(m-n\right)\left(m+z\right)} \\ &= 2\sum_{\substack{m,n=0\\ m\neq n}}^{N} \frac{(-1)^{m+n}}{\left(m-n\right)\left(n+z\right)} = 2\sum_{n=0}^{N} \sum_{\substack{m=0\\ m\neq n}}^{N} \frac{(-1)^n}{n+z} \frac{(-1)^m}{m-n} \\ &= 2\sum_{n=0}^{N} \left(\frac{(-1)^n}{n+z} \sum_{\substack{m=0\\ m\neq n}}^{N} \frac{(-1)^m}{m-n}\right). \end{aligned}$$

Letting $j = m - n$,

$$\sum_{\substack{m=0 \\ m \neq n}}^{N} \frac{(-1)^m}{m-n} = \sum_{j=1}^{N-n} \frac{(-1)^{j+n}}{j} \leq \frac{1}{N}.$$

Then

$$\begin{aligned} A_N^2 - B_N &= 2\sum_{n=0}^{N} \left(\frac{(-1)^n}{n+z} \sum_{j=1}^{N-n} \frac{(-1)^{j+n}}{j} \right) \\ &= 2\sum_{n=0}^{N} \left(\frac{1}{n+z} \sum_{j=1}^{N-n} \frac{(-1)^{j}}{j} \right) \\ &\leq 2\sum_{n=0}^{N} \left(\frac{1}{N(n+z)} \right) \leq \frac{2}{N} \log(N+z). \end{aligned}$$

The bounding term goes to 0 as $N \to \infty$. Hence,

$$\lim_{N\to\infty} \left[A_N^2 - B_N\right] = 0,$$

and the result follows. ■

Proposition 3. *Let $a, b \in \mathbb{N}$ and let $s \in \mathbb{C}$ with $\Re(s) \geq 1$. Then*

$$\frac{\partial}{\partial b}[D_{\text{Choi}}(s,a,b)] = -sD_{\text{Choi}}(s+1,a,b). \tag{9}$$

Proof. The proof is straightforward, but we include it here for completeness. To justify differentiation, note that $D_{\text{Choi}}(s,a,b)$ is convergent for $\Re(s) \geq 1$. Write

$$\begin{aligned} \frac{\partial}{\partial b}[D_{\text{Choi}}(s,a,b)] &= \frac{\partial}{\partial b} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(an+b)^s} + \frac{(-1)^{s-1}}{(an+a-b)^s} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{-s}{(an+b)^{s+1}} + \frac{(-1)^{s-1}s}{(an+a-b)^{s+1}} \right) \\ &= -s\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(an+b)^{s+1}} + \frac{(-1)^{s}}{(an+a-b)^{s+1}} \right) \\ &= -sD_{\text{Choi}}(s+1,a,b), \end{aligned}$$

as required. ■

Proposition 3 suggests that we can differentiate with respect to b to obtain a formula for $D_{\text{Choi}}(s+1,a,b)$ if we know the exact formula for $D_{\text{Choi}}(s,a,b)$. Proposition 2 suggests that we can obtain a formula for $D_{\text{Choi}}(1,a,b)$ if we know the exact formula for $D(2,a,b)$. Hence, this describes an inductive process where we differentiate with respect to b to obtain successive formulas.

Using the methods of Balanzario [2],

$$D(2,a,b) = \frac{2\pi^2}{a}\sum_{j=1}^{a}\cos\left(\frac{2\pi jb}{a}\right)\left(\frac{j^2}{a^2} - \frac{j}{a} + \frac{1}{6}\right). \tag{10}$$

To compute (10), we consider the formulas of sums of trigonometric functions in arithmetic progression given by Knapp [12],

$$\sum_{k=1}^{n-1}\sin(a+kd) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}\sin\left(a + \frac{(n-1)d}{2}\right) \tag{11}$$

and

$$\sum_{k=1}^{n-1}\cos(a+kd) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}\cos\left(a + \frac{(n-1)d}{2}\right). \tag{12}$$

To compute the formula of linear sums of cosines in arithmetic progression, we take the partial derivative of (11) with respect to d. This is shown in the following lemma.

Lemma 4. *For $a, d \in \mathbb{R}$, $d \neq 0$, and $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n-1} k\cos(a+kd) = \frac{n\sin\left(a + \left(n-\frac{1}{2}\right)d\right)}{2\sin\left(\frac{d}{2}\right)} - \frac{\sin\left(\frac{nd}{2}\right)\sin\left(a+\frac{nd}{2}\right)}{2\sin^2\left(\frac{d}{2}\right)}. \tag{13}$$

Proof. We see that

$$\sum_{k=1}^{n-1} k\cos(a+kd) = \frac{\partial}{\partial d}\left[\sum_{k=1}^{n-1}\sin(a+kd)\right].$$

Differentiating with respect to d yields the result. ∎

To compute the formula of quadratic sums of cosines in arithmetic progression, we take the second partial derivative with respect to d. We show this in the next lemma.

Lemma 5. *For $a, d \in \mathbb{R}$, $d \neq 0$, and $n \in \mathbb{N}$,*

$$\begin{aligned}\sum_{k=1}^{n-1} k^2\cos(a+kd) = {} & \frac{n^2\sin\left(a + \left(n-\frac{1}{2}\right)d\right)}{2\sin\left(\frac{d}{2}\right)} + \frac{n\cos(a+nd)}{2\sin^2\left(\frac{d}{2}\right)} \\ & - \frac{\cos\left(\frac{d}{2}\right)\sin\left(\frac{nd}{2}\right)\cos\left(a+\frac{nd}{2}\right)}{2\sin^3\left(\frac{d}{2}\right)}. \end{aligned} \tag{14}$$

Proof. We see that

$$\sum_{k=1}^{n-1} k^2\cos(a+kd) = -\frac{\partial^2}{\partial d^2}\left[\sum_{k=1}^{n-1}\cos(a+kd)\right].$$

Differentiating with respect to d twice yields the result. ∎

We now apply Lemmas 4 and 5 to (10) to obtain the following theorem.

Theorem 6. *Let $a, b \in \mathbb{N}$. Then*

$$D(2, a, b) = \frac{\pi^2}{a^2} \csc^2\left(\frac{b\pi}{a}\right). \tag{15}$$

Proof. We decompose (10) into three sums. We invoke (12) to compute

$$\begin{aligned}
\sum_{j=1}^{a} \cos\left(\frac{2\pi jb}{a}\right) &= \sum_{j=0}^{a-1} \cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) \\
&= 0.
\end{aligned}$$

Next, we use (13) to compute

$$\begin{aligned}
\sum_{j=1}^{a} j\cos\left(\frac{2\pi jb}{a}\right) &= \sum_{j=0}^{a-1} (j+1)\cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) \\
&= \sum_{j=0}^{a-1} j\cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) + \sum_{j=0}^{a-1} \cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) \\
&= \frac{a}{2}.
\end{aligned}$$

Finally, we use (14) to compute

$$\begin{aligned}
\sum_{j=1}^{a} j^2\cos\left(\frac{2\pi jb}{a}\right) &= \sum_{j=0}^{a-1} (j+1)^2\cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) \\
&= \sum_{j=0}^{a-1} j^2\cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) + \sum_{j=0}^{a-1} 2j\cos\left(\frac{2b\pi}{a} + \frac{2b\pi}{a}j\right) \\
&= \frac{a^2}{2} + \frac{a}{2}\csc^2\left(\frac{b\pi}{a}\right).
\end{aligned}$$

Combining these three sums and multiplying by $2\pi^2/a$ gives (15). ■

The next two results are now immediate.

Theorem 7. *Let $a, b \in \mathbb{N}$. Then*

$$D_{\text{Choi}}(1, a, b) = \frac{\pi}{a} \csc\left(\frac{b\pi}{a}\right). \tag{16}$$

Proof. The result follows from (7) applied to the previous theorem. ■

Theorem 8. *Let $a, b \in \mathbb{N}$. Then*

$$D_{\text{Choi}}(2,a,b) = \frac{\pi^2}{a^2}\csc\left(\frac{b\pi}{a}\right)\cot\left(\frac{b\pi}{a}\right) = \frac{\pi^2}{a^2}\csc^2\left(\frac{b\pi}{a}\right)\cos\left(\frac{b\pi}{a}\right). \tag{17}$$

Proof. The result follows from (9) applied to the previous theorem. ■

Theorem 8 verifies Choi's five values [3]. We continually apply Proposition 3 to prove Theorem 1.

Proof of Theorem 1. We prove using induction. Our base case comes from Theorem 7, where (16) can be expressed as

$$D_{\text{Choi}}(1,a,b) = \frac{\pi}{a}\csc\left(\frac{b\pi}{a}\right)P_0\left(\cos\left(\frac{b\pi}{a}\right)\right),$$

where $P_1(x) = 1$. Then assume (3) to hold. We need to show $D_{\text{Choi}}(n+1,a,b)$ holds. To do this, let $X = b\pi/a$ for brevity and use Proposition 3 and (9) to get

$$D_{\text{Choi}}(n+1,a,b) = -\frac{1}{n}\frac{\partial}{\partial b}\left[D_{\text{Choi}}(n,a,b)\right] = \frac{\pi^{n+1}}{a^{n+1}n!}\csc^{n+1}(X)\,\mathcal{P},$$

where

$$\begin{aligned}
\mathcal{P} &= \sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor} T(n-1,k)\left(n\cos^{n-2k}(X) + (n-1-2k)\sin^2(X)\cos^{n-2-2k}(X)\right) \\
&= \sum_{k=0}^{\lfloor\frac{n+3}{2}\rfloor} T(n-1,k)\left(n\cos^{n-2k}(X) + (n-1-2k)\sin^2(X)\cos^{n-2-2k}(X)\right) \\
&= \sum_{k=0}^{\lfloor\frac{n+3}{2}\rfloor} T(n-1,k)\left((2k+1)\cos^{n-2k}(X) + (n-1-2k)\cos^{n-2-2k}(X)\right) \\
&= \sum_{k=0}^{\lfloor\frac{n+3}{2}\rfloor} T(n-1,k)(2k+1)\cos^{n-2k}(X) \\
&\quad + \sum_{k=0}^{\lfloor\frac{n+3}{2}\rfloor} T(n-1,k)(n-1-2k)\cos^{n-2-2k}(X).
\end{aligned}$$

For the first sum, we take the first term of the sum to obtain

$$\begin{aligned}
\sum_{k=0}^{\lfloor\frac{n+3}{2}\rfloor} T(n-1,k)(2k+1)\cos^{n-2k}(X) &= \cos^n(X) \\
&\quad + \sum_{k=1}^{\lfloor\frac{n+3}{2}\rfloor} T(n-1,k)(2k+1)\cos^{n-2k}(X).
\end{aligned}$$

Then we apply the recursion (4) and reabsorb the first term of the sum

$$\cos^n(X) + \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} T(n-1,k)(2k+1)\cos^{n-2k}(X) = \sum_{k=0}^{\lfloor \frac{n+3}{2} \rfloor} T(n,k)\cos^{n-2k}(X)$$
$$- \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} T(n-1,k-1)(n-2k+1)\cos^{n-2k}(X).$$

However, these two sums are equivalent because of reindexing:

$$\sum_{k=0}^{\lfloor \frac{n+3}{2} \rfloor} T(n-1,k)(n-1-2k)\cos^{n-2-2k}(X)$$
$$= \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} T(n-1,k-1)(n-2k+1)\cos^{n-2k}(X).$$

Thus, adding the sums together produces

$$\mathcal{P} = \sum_{k=0}^{\lfloor \frac{n+3}{2} \rfloor} T(n,k)\cos^{n-2k}(X) = P_n(\cos(X)).$$

Thus,

$$D_{\text{Choi}}(n+1,a,b) = \frac{\pi^{n+1}}{a^{n+1}n!}\csc^{n+1}\left(\frac{b\pi}{a}\right) P_n\left(\cos\left(\frac{b\pi}{a}\right)\right)$$
$$= \frac{\pi^{n+1}}{a^{n+1}n!}\csc^{n+1}\left(\frac{b\pi}{a}\right) \sum_{k=0}^{\lfloor \frac{n+3}{2} \rfloor} T(n,k)\cos^{n-2k}\left(\frac{b\pi}{a}\right),$$

as required. ■

Note that for n even, $P_n(x)$ has no constant term, so $P_n(0) = 0$. We now provide a corollary immediately applying this property.

Corollary 1. *Let $n, a \in \mathbb{N}$ be even. If $b \equiv a/2 \pmod{a}$, then*

$$D_{\text{Choi}}(n+1,a,b) = 0.$$

Proof. Suppose $b \equiv a/2 \pmod{a}$. Then $b - a/2$ is a multiple of a, so there exists some $k \in \mathbb{N}$ such that $b - a/2 = ka$. Then $b/a = k + 1/2$. Hence,

$$\cos\left(\frac{b\pi}{a}\right) = \cos\left(\left(k+\frac{1}{2}\right)\pi\right) = 0 \quad \text{so} \quad P_n\left(\frac{b\pi}{a}\right) = 0.$$

The result immediately follows from Theorem 1 and (3). ■

4 Special Values of Choi Dirichlet Series

Here, we prove Theorem 2, the general formula that calculates special values of $D(s, a, b)$ on positive integers. Similar to the methods of the previous section, we use the following proposition.

Proposition 4. *Let $a, b \in \mathbb{N}$ and let $s \in \mathbb{C}$ with $\Re(s) \geq 1$. Then*

$$\frac{\partial}{\partial b}[D(s, a, b)] = -sD(s + 1, a, b). \tag{18}$$

Proof. The proof is straightforward and similar to that of Proposition 3. To justify differentiation, note that $D(s, a, b)$ is convergent for $\Re(s) \geq 1$. Write

$$\begin{aligned}
\frac{\partial}{\partial b}[D(s, a, b)] &= \frac{\partial}{\partial b} \sum_{n=0}^{\infty} \left(\frac{1}{(an + b)^s} + \frac{(-1)^s}{(an + a - b)^s} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{-s}{(an + b)^{s+1}} + \frac{(-1)^s s}{(an + a - b)^{s+1}} \right) \\
&= -s \sum_{n=0}^{\infty} \left(\frac{1}{(an + b)^{s+1}} + \frac{(-1)^{s+1}}{(an + a - b)^{s+1}} \right) \\
&= -sD(s + 1, a, b),
\end{aligned}$$

as required. ■

Thus, using Proposition 4, we inductively derive new formulas for successive $D(s, a, b)$. This gives us the proof of Theorem 2.

Proof of Theorem 2. We prove using induction. Our base comes from Theorem 6, where (15) can be expressed as

$$D(2, a, b) = \frac{\pi^2}{a^2} \csc^2\left(\frac{b\pi}{a}\right) = \frac{\pi^2}{a^2} \csc^2\left(\frac{b\pi}{a}\right) Q_1\left(\frac{b\pi}{a}\right),$$

where $Q_1(x) = 1$. Then assume (15) to hold. We need to show $D(n + 1, a, b)$ holds. To do this, let $X = b\pi/a$ for brevity and use Proposition 4 and (18) to get

$$D(n + 1, a, b) = -\frac{1}{n}\frac{\partial}{\partial b}[D(n, a, b)] = \frac{\pi^{n+1}}{a^{n+1} n!} \csc^{n+1}(X)\mathcal{Q},$$

where

$$
\begin{aligned}
\mathcal{Q} &= \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} t(n-1,k)\left(n\cos^{n-2k+1}(X) + (n-2k)\sin^2(X)\cos^{n-2k-1}(X)\right) \\
&= \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} t(n-1,k)\left(2k\cos^{n-2k+1}(X) + (n-2k)\cos^{n-2k-1}(X)\right) \\
&= \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} t(n-1,k)(2k)\cos^{n-2k+1}(X) \\
&\quad + \sum_{k=2}^{\lfloor \frac{n+3}{2} \rfloor} t(n-1,k-1)(n-2k+2)\cos^{n-2k+1}(X) \\
&= \sum_{k=2}^{\lfloor \frac{n+3}{2} \rfloor} \left(2kt(n-1,k) + (n-2k+2)t(n-1,k-1)\right)\cos^{n-2k+1}(X) \\
&\quad + 2t(n-1,1)\cos^{n-1}(X).
\end{aligned}
$$

For the sum, use the recursion (6) to obtain

$$
\sum_{k=2}^{\lfloor \frac{n+3}{2} \rfloor} \left(2kt(n-1,k) + (n-2k+2)t(n-1,k-1)\right)\cos^{n-2k+1}(X) = \sum_{k=2}^{\lfloor \frac{n+3}{2} \rfloor} t(n,k)\cos^{n-2k+1}(X).
$$

Thus, we have

$$
\begin{aligned}
\mathcal{Q} &= \sum_{k=2}^{\lfloor \frac{n+3}{2} \rfloor} t(n,k)\cos^{n-2k+1}(X) + 2t(n-1,1)\cos^{n-1}(X) \\
&= \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} t(n,k)\cos^{n-2k+1}(X) + 2t(n-1,1)\cos^{n-1}(X) - t(n,1)\cos^{n-1}(X)
\end{aligned}
$$

and once again we use recursion (6), namely $t(n,1) = 2^{n-1}$, to obtain

$$
\mathcal{Q} = \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} t(n,k)\cos^{n-2k+1}(X).
$$

Thus,

$$\begin{aligned}
D(n+1,a,b) &= \frac{\pi^{n+1}}{a^{n+1}n!}\csc^{n+1}\left(\frac{b\pi}{a}\right)Q_n\left(\cos\left(\frac{b\pi}{a}\right)\right)\\
&= \frac{\pi^{n+1}}{a^{n+1}n!}\csc^{n+1}\left(\frac{b\pi}{a}\right)\sum_{k=1}^{\lfloor\frac{n+3}{2}\rfloor} t(n,k)\cos^{n-2k+1}\left(\frac{b\pi}{a}\right).
\end{aligned}$$

Now we need to prove the $n=1$ case. We use Proposition 4 and write

$$\begin{aligned}
D(1,a,b) &= -\int D(2,a,b)\,db\\
&= -\int \frac{\pi^2}{a^2}\csc^2\left(\frac{b\pi}{a}\right)\,db\\
&= \frac{\pi}{a}\cot\left(\frac{b\pi}{a}\right)\\
&= \frac{\pi}{a}\csc\left(\frac{b\pi}{a}\right)\cos\left(\frac{b\pi}{a}\right)\\
&= \frac{\pi}{a}\csc\left(\frac{b\pi}{a}\right)Q_0\left(\cos\left(\frac{b\pi}{a}\right)\right),
\end{aligned}$$

as required. ■

Note that for n even, $Q_n(x)$ has no constant term, so $Q_n(0)=0$. We now provide a corollary immediately applying this property.

Corollary 2. *Let $n, a \in \mathbb{N}$ be even. If $b \equiv a/2 \pmod{a}$, then*

$$D(n+1,a,b) = 0.$$

Proof. Suppose $b \equiv a/2 \pmod{a}$. Then $b - a/2$ is a multiple of a, so there exists some $k \in \mathbb{N}$ such that $b - a/2 = ka$. Then $b/a = k + 1/2$. Hence,

$$\cos\left(\frac{b\pi}{a}\right) = \cos\left(\left(k+\frac{1}{2}\right)\pi\right) = 0 \quad \text{so} \quad Q_n\left(\frac{b\pi}{a}\right) = 0.$$

The result immediately follows from Theorem 2 and (5). ■

5 Conclusion and Future Work

We found closed formulas for special values of Choi Alternating Dirichlet series and Choi Dirichlet series. The closed formulas for these Dirichlet series relied on taking repeated partial derivatives with respect to b, generating an inductive process to prove the general formulas. The closed formulas verify the exact values that Choi obtained. These Dirichlet series were studied by Choi and Shimura, among others. Moreover, we proved the existence of a meromorphic continuation of the Choi Alternating Dirichlet series, expressible as a sum of two Lerch zeta functions, which have meromorphic continuations.

Future work include finding a functional equation for these Choi Dirichlet series as well as relating these series to multiple zeta functions. Another potential focus is to explore the possibility of some form of Euler product associated with the Choi Alternating Dirichlet series. Provided one exists, future work includes finding more properties about prime numbers.

6 Acknowledgements

I thank my mentor, Dr. Dominic Lanphier of the Department of Mathematics at Western Kentucky University, for his much valued mentorship on this project. I also give my thanks to my committee members, Dr. Dominic Lanphier, Dr. Tom Richmond, and Dr. Samangi Munasinghe. Finally, I give my gratitude to the Gatton Academy of Mathematics and Science for providing funding to this project.

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