THE ZERO-FREE REGION OF THE RIEMANN ZETA FUNCTION AND BEURLING ZETA FUNCTIONS

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1. Introduction

The Riemann zeta function is one of the most fundamental mathematical objects used to study prime numbers, their behavior, and their distribution. For $s \in \mathbb{C}$ with $\Re(s) > 1$, the zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Some evaluations of $\zeta(s)$ are well known; Euler [7] proved a general closed form for special values at the positive even integers such as $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. For other $s \in \mathbb{C}$, $\zeta(s)$ is defined by its analytical continuation [20] with a pole at $s = 1$. One of its most defining properties is its Euler product: For any $s \in \mathbb{C}$ and $\Re(s) > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - 1/p^s}, \tag{1}$$

where the product is absolutely convergent. By analyzing the behavior of $\zeta(s)$ when $s \to 1$, the pole at $s = 1$ immediately implies the infinitude of primes.

Beurling [3] developed a more generalized number system of prime number theory. He considered a sequence $\mathscr{P} = \{\lambda_j\}_{j \in \mathbb{N}}$ of increasing real numbers $1 < \lambda_1 \leq \lambda_2 \leq \cdots$ such that $\lambda_j \to \infty$; here $\mathscr{P}$ is the set of *generalized primes*. He then constructed the set of *generalized integers* $\mathscr{N}$ consisting of all finite products of the form $\lambda_1^{k_1} \cdots \lambda_r^{k_r}$ and defined the generalized zeta function

$$\zeta_B(s) = \sum_{\mathbf{k}} \frac{1}{\left(\lambda_1^{k_1} \lambda_2^{k_2} \cdots\right)^s},$$

where $\mathbf{k} = (k_1, k_2, \ldots)$ is a vector with nonnegative integer coordinates where all but a finite number of which are zero. Importantly, $\zeta_B(s)$ also admits an Euler product

$$\zeta_B(s) = \prod_{\lambda \in \mathscr{P}} \frac{1}{1 - 1/\lambda^s}.$$

This paper focuses on the connection between $\zeta(s)$ and $\zeta_B(s)$, notably their implications on the distribution of prime numbers in analytical number theory and Fourier analysis.

Fourier analytic-tools are effective in deducing properties about $\zeta(s)$, such as the location of its zeroes, which imply further properties about primes. For example, one of the quintessential achievements in analytical number theory is the prime number theorem, which states that as $x \to \infty$, the number of primes $\pi(x)$ less than a positive real number x is (see [21])

$$\pi(x) = \int_2^x \frac{dt}{\log t} + o(1). \tag{2}$$

While Gauss and Legendre conjectured (2) based on numerical data, the proof of the prime number theorem eluded mathematicians for about a century until Hadamard [9] and de la Vallée Poussin [18] independently proved it by using the fact that there are no zeroes of $\zeta(s)$ on the line $\{s \in \mathbb{C} \mid \Re(s) = 1\}$, i.e., $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R} \setminus \{0\}$. This zero-free region extends the trivial zero-free region $\{s \in \mathbb{C} \mid \Re(s) > 1\}$ given by (1) since the product is absolutely convergent for such s and all terms in the product are nonzero. This extension implies the prime number theorem. In fact, the location of the nontrivial zeroes imply numerous properties about primes. In 1899, de la Vallée Poussin [19] extended the zero-free region to the *classical zero-free region* as follows.

Theorem 1. *There exists an absolute constant $C > 0$ such that the region*

$$\left\{ s \in \mathbb{C} \;\middle|\; \Re(s) > 1 - \frac{C}{\log(\Im(s))} \text{ and } \Im(s) \geq 2 \right\}$$

contains no zeroes of $\zeta(s)$.

This improved the error bound in (2) to the following.

Theorem 2. *For $x \geq 2$, there exists an absolute constant $C > 0$ such that*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{\log x} \right) \right).$$

The Riemann hypothesis famously states that all the nontrivial zeroes lie on the critical line $\{s \in \mathbb{C} \mid \Re(s) = 1/2\}$. The truth of this conjecture would imply the best possible bounds in the prime number theorem, the growth of arithmetical functions, and other number-theoretic objects. Major progress has been made towards extending the zero-free region towards the critical line, with contributions over the past century from Littlewood [14], Chudakov [4], Korobov [10], Vinogradov [22], and Ford [8]. The current best zero-free region is

$$\left\{ s \in \mathbb{C} \;\middle|\; \Re(s) \geq 1 - \frac{1}{53.989(\log \Im(s))^{2/3}(\log\log \Im(s))^{1/3}} \text{ and } \Im(s) \geq 3 \right\},$$

due to Bellotti [2] in 2024.

In the generalized setting of Beurling primes, the zero-free region of $\zeta_B(s)$ give similar implications on the distribution of generalized primes. Let $N_B(x)$ denote the number of generalized integers in $\mathscr{N}$ less than x counted with multiplicity, and let the generalized prime counting function $\pi_B(x)$ be the number of Beurling primes $\lambda_j \in \mathscr{P}$ less than x. We have the following analogous results of the classical zero-free region and the prime number theorem under a certain hypothesis.

Theorem 3. *Suppose that $N_B(x) = \kappa x + O\left(x^\theta\right)$ for some $\kappa > 0$ and $0 \le \theta < 1$. Then there exists an absolute constant $C > 0$ such that the region*

$$\left\{ s \in \mathbb{C} \;\middle|\; \Re(s) > 1 - \frac{C(1-\theta)}{\log(\Im(s))} \text{ and } \Im(s) \ge 2 \right\}$$

contains no zeroes of $\zeta_B(s)$.

Theorem 4. *Suppose that $N_B(x) = \kappa x + O\left(x^\theta\right)$ for some $\kappa > 0$ and $0 \le \theta < 1$. Then there exists an absolute constant $C > 0$ such that for all $x \ge 2$,*

$$\pi_B(x) = \int_2^\infty \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right).$$

In this paper, we give a brief survey of the proofs of the classical zero-free regions of both $\zeta(s)$ and $\zeta_B(s)$ using Fourier analytic-tools on discrete measures. The organization of this paper is as follows. In Section 2, we present the original proof of the classical zero-free region, followed by a proof of the improvement presented Theorem 2 in Section 3. In Section 4, we give a brief discussion about the properties of the Beurling generalized number systems and the generalization of Fourier analytic-methods to sketch a proof of the zero-free region given by Theorem 3 in this general setting. Finally, in Section 5 we give a proof of Theorem 4 under the above zero-free region and a brief discussion of its optimality by Diamond, Montgomery, and Vorhauer [5].

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2. The Classical Zero-Free Region of $\zeta(s)$

In this section, we give a brief overview of the properties of $\zeta(s)$, such as its analytical continuation, and then prove Theorem 1. The proof of Theorem 1 relies on the analytical continuation of $\zeta(s)$ as well as the Riemann-von Mangoldt formula, which gives an expression relating $\zeta'(s)/\zeta(s)$ to the nontrivial zeroes of $\zeta(s)$. This formula, given by Proposition 1, is useful as it provides a more direct analysis of the nontrivial zeroes, which is pertinent to our study of the classical zero-free region of $\zeta(s)$. Here, Γ denotes the Gamma function.

Proposition 1. *There exists some constant $B > 0$ such that for all $s \neq 1$,*

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s} - \frac{1}{s-1} + \frac{\log \pi}{2} - \frac{\Gamma'(s/2)}{2\Gamma(s/2)} + \sum_{\text{zeroes}\,\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

We give the proof of Proposition 1 in Section 2.2. In Section 2.1, we first give the analytical continuation and the functional equation of $\zeta(s)$ as well as another identity for $\zeta'(s)/\zeta(s)$ using the von Mangoldt function. Finally, in Section 2.3, we combine these tools to prove the classical zero-free region.

2.1. **The Logarithmic Derivative.** For any $x > 0$, let δ_x denote the Dirac measure at x. For any two measures μ and ν, let $\mu * \nu$ denote the convolution of μ and ν, and let μ^{*n} denote the n-fold convolution $\mu * \cdots * \mu$. We also define the pushforward exponential and logarithm of μ by

$$\exp_*(\mu) = \delta_0 + \sum_{n=1}^{\infty} \frac{\mu^{*n}}{n!}, \qquad \log_*(\delta_0 + \mu) = \sum_{n=1}^{\infty} (-1)^n \frac{\mu^{*n}}{n},$$

so that

$$\mathop{\Huge *}_{j \in J} \mu_j = \exp_*\left(\sum_{j \in J} \log_*(A_j) \right).$$

For any function μ, define the Laplace-Fourier transform $\mathcal{L}\{\mu\}(s)$ by

$$\mathcal{L}\{\mu\}(s) = \int_0^\infty e^{-st}\, d\mu(t),$$

whereas its Fourier transform is given by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} \mu(t)\, dt.$$

Definition 1. The *von Mangoldt function* $\Lambda(n)$ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some integer } k \geq 1 \text{ and some prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

The *von Mangoldt measure* $\mu(t)$ is defined by

$$\mu(t) = \sum_{p\,\text{prime}} \Lambda(t) \sum_{r=1}^{\infty} \delta_{r \log p}(t).$$

Specifically, we prove $-\zeta'(s)/\zeta(s) = \mathcal{L}\{\mu\}(s)$ for $\Re(s) > 1$. First, define the discrete measure N by

$$N = \sum_{n \in \mathbb{N}} \delta_{\log n}.$$

When $\Re(s) > 1$, by the monotone convergence theorem and Fubini's theorem we may switch the summation and integral to obtain

(3)

$$\mathcal{L}\{N\}(s) = \int_0^\infty e^{-st}\, dN(t) = \int_0^\infty \sum_{n \in \mathbb{N}} e^{-st}\, d\delta_{\log n}(t) = \sum_{n \in \mathbb{N}} \int_0^\infty e^{-st}\, d\delta_{\log n}(t) = \zeta(s).$$

We now give a meromorphic extension of $\zeta(s)$ to the region $\{s \in \mathbb{C} \mid \Re(s) > 0\}$.

Lemma 1. *For* $\Re(s) > 0$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \{x\} x^{-s-1}\, dx. \tag{4}$$

Proof. Note that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = s \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1}\, dx = s \int_1^{\infty} [x] x^{-s-1}\, dx$$

where $[x]$ denotes the integer part of x. Writing $x = [x] + \{x\}$, where $\{x\}$ denotes the fractional part of x, we obtain

$$\zeta(s) = s \int_1^{\infty} (x - \{x\}) x^{-s-1}\, dx = \frac{s}{s-1} - s \int_1^{\infty} \{x\} x^{-s-1}\, dx,$$

where the last integral converges for $\Re(s) > 0$. □

Using analytical continuation, $\zeta(s)$ is extended to the entire complex plane $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies the functional equation below.

Proposition 2. *The function $\zeta(s)$ satisfies the functional equation*

$$\pi^{-s/2} \Gamma \left(\frac{s}{2} \right) \zeta(s) = \pi^{-(1-s)/2} \Gamma \left(\frac{1-s}{2} \right) \zeta(1-s). \tag{5}$$

In fact, (5) implies that the only zeroes of $\zeta(s)$ in the half plane $\{s \in \mathbb{C} \mid \Re(s) < 0\}$ are precisely of the set $\{s = -2k \mid k \in \mathbb{N}\}$. These zeroes are simple and are referred to as the *trivial zeroes*. Since $\zeta(s) \neq 0$ for $\Re(s) > 1$, the *nontrivial zeroes* are located in the region $\{s \in \mathbb{C} \mid 0 \leq \Re(s) \leq 1\}$. We refer the reader to Davenport [6] for details.

To prove Proposition 2, we introduce the theta function $\theta(x)$ and show that it satisfies the functional equation $\theta(1/x) = x^{1/2}\theta(x)$.

Definition 2. For all $x > 0$, the *theta function* $\theta(x)$ is given by

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}.$$

Lemma 2. *The function $\theta(x)$ satisfies the functional equation $\theta(1/x) = x^{1/2}\theta(x)$.*

Proof. Let $f(y) = e^{-\pi x y^2}$, so its Fourier transform is $\widehat{f}(\xi) = x^{-1/2} e^{-\pi \xi^2 / x}$. The Poisson summation formula gives

$$\theta(x) = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / x} = \frac{\theta(1/x)}{\sqrt{x}},$$

as required. □

With Lemma 2 we now prove Proposition 2.

Proof of Proposition 2. With the substitution $t = \pi n^2 x$ in the integral we have

$$\pi^{-s/2} \Gamma(s/2) n^{-s} = \pi^{-s/2} n^{-s} \int_0^{\infty} e^{-t} t^{s-1}\, dt = \int_0^{\infty} e^{-\pi n^2 x} x^{s/2-1}\, dx.$$

The integral converges absolutely for $\Re(s) > 1$, so we sum over all positive integers $n \in \mathbb{N}$ and switch the summation and integral by Fubini's theorem to obtain

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty x^{s/2-1} \sum_{n=1}^\infty e^{-\pi n^2 x}\, dx = \int_0^\infty x^{s/2-1}\omega(x)\, dx, \tag{6}$$

where $\omega(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$. Note that $2\omega(x) = \theta(x) - 1$, so by Lemma 2 we have

$$\omega(1/x) = -\frac{1}{2} + \frac{x^{1/2}}{2} + x^{1/2}\omega(x). \tag{7}$$

We split the integral in (6) at $x = 1$, giving

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^1 x^{s/2-1}\omega(x)\, dx + \int_1^\infty x^{s/2-1}\omega(x)\, dx \\
&= \int_1^\infty x^{-s/2-1}\omega(1/x) + \int_1^\infty x^{s/2-1}\omega(x)\, dx \\
&= \int_1^\infty x^{-s/2-1}\left(-\frac{1}{2} + \frac{x^{1/2}}{2} + x^{1/2}\omega(x)\right) dx + \int_1^\infty x^{s/2-1}\omega(x)\, dx, \\
&= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left(x^{s/2-1} + x^{(1-s)/2-1}\right)\omega(x)\, dx.
\end{aligned}$$

where in the first integral we make the substitution $x \mapsto 1/x$ and used (7). The result follows since the last line is invariant under $s \mapsto 1 - s$. $\square$

With Proposition 2 proved, we conclude by showing that $-\zeta'(s)/\zeta(s) = \mathcal{L}\{\mu\}(s)$ for $\Re(s) > 1$. For every prime p, define the measures N_p by

$$N_p = \sum_{r=0}^\infty \delta_{\log p^r},$$

which has pushforward

$$\log_*(N_p) = \log_*\left(\sum_{r=0}^\infty \delta_{r\log p}\right) = \log_*\left(\sum_{r=0}^\infty \delta_{\log p}^{*r}\right) = \sum_{r=1}^\infty \frac{\delta_{r\log p}}{r}$$

Furthermore, we define the measure P by

$$P = \sum_{p\,\text{prime}} \log_*(N_p) = \sum_{p\,\text{prime}}\sum_{r=1}^\infty \frac{\delta_{r\log p}}{r}.$$

Then the fundamental theorem of arithmetic implies that

$$N = \mathop{\Huge *}_{p\,\text{prime}} N_p = \exp_*\left(\sum_{p\,\text{prime}} \log_*(N_p)\right) = \exp_*(P). \tag{8}$$

This identity (8) implies the following proposition, choosing a suitable branch of the complex logarithm.

Lemma 3. *For all $\Re(s) > 1$, $\log \mathcal{L}\{N\}(s) = \mathcal{L}\{P\}(s)$.*

Proof. For $\Re(s) > 1$, we have $\mathcal{L}\{N\}(s) = \mathcal{L}\{\exp_*(P)\}(s) = e^{\mathcal{L}\{P\}(s)}$ from (8). $\square$

Lemma 3 with (3) implies $\log\zeta(s) = \mathcal{L}\{P\}(s)$ for $\Re(s) > 1$. We now use this to prove the following.

Proposition 3. *For* $\Re(s) > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \mathcal{L}\{\mu\}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof. For $\Re(s) > 1$, we switch the summation and integral by the monotone convergence theorem and Fubini's theorem to obtain

$$\mathcal{L}\{tP\}(s) = \int_0^\infty e^{-st}\, d(tP(t)) = \sum_{p\,\text{prime}} \sum_{r=1}^{\infty} \frac{1}{r} \int_0^\infty te^{-st}\, d\delta_{r\log p}(t) = \sum_{p\,\text{prime}} \log p \sum_{r=1}^{\infty} p^{-rs}$$

whereas we also have

$$\mathcal{L}\{\mu\}(s) = \int_0^\infty e^{-st}\, d\mu(t) = \sum_{p\,\text{prime}} \sum_{r=1}^{\infty} \int_0^\infty \Lambda(t) e^{-st}\, d\delta_{r\log p}(t) = \sum_{p\,\text{prime}} \log p \sum_{r=1}^{\infty} p^{-rs}.$$

Thus, $\mathcal{L}\{\mu\}(s) = \mathcal{L}\{tP\}(s)$ for $\Re(s) > 1$. Then the identity

$$\mathcal{L}\{tP\}(s) = -\frac{d}{ds}\mathcal{L}\{P\}(s) = -\frac{d}{ds}\log\zeta(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

completes the first equality. For the second equality, note that

$$\sum_{p\,\text{prime}} \log p \sum_{r=1}^{\infty} p^{-rs} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

as required. $\square$

2.2. **The Riemann-von Mangoldt Formula.** We prove the Riemann-von Mangoldt formula. Recall that an entire function $f(s)$ has *finite order* if there exists some number $\alpha > 0$ such that $f(s) = O\left(e^{|s|^\alpha}\right)$ as $|s| \to \infty$, and the *order* of $f(s)$ is defined to be the infimum of all such αs. An entire function of order 1 satisfies the following characterization regarding its zeroes. (See [1] or [6].)

Theorem 5. *If* $f\colon \mathbb{C} \to \mathbb{C}$ *be an entire function of order* 1*, then there exist constants* $A, B > 0$ *such that*

$$f(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\rho_n}\right) e^{s/\rho_n},$$

where the ρ_n *are the zeroes of* $f(s)$.

We cannot apply Theorem 5 directly to $\zeta(s)$ because it is not entire. Rather, we apply the theorem to the Riemann xi function given by $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ after showing that $\xi(s)$ is entire and is of order 1. The fact that $\xi(s)$ is entire follows from the fact that it has no poles for $\Re(s) \geq 1/2$ and is an even function of $s - 1/2$, since $\xi(s) = \xi(1-s)$ follows from (5).

To show that $\xi(s)$ is of order 1, we show that there exists some constant $C > 0$ such that for $\Re(s) \geq 1/2$, we have

$$|\xi(s)| < \exp(C|s|\log|s|); \tag{9}$$

from here, the identity $\xi(s) = \xi(1-s)$ then implies the result. There exist some constant $C_1 > 0$ such that $\left|s(s-1)\pi^{-s/2}\right| < \exp(C_1|s|)$ and $|\Gamma(s/2)| < \exp(C_1|s|\log|s|)$ by Stirling's formula, so it remains to bound $|\zeta(s)|$. But using (4), there exists some constant $C_2 > 0$ such that

$$|\zeta(s)| = \left|\frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}\,dx\right| < C_2|s|$$

for sufficiently large $|s|$. Combining all three bounds gives (9).

The definition of $\xi(s)$ has the convenient property that the zeroes of $\xi(s)$ are exactly the nontrivial zeroes of $\zeta(s)$. This follows since each simple zero of $\zeta(s)$ at $s = -2k$ for some $k \in \mathbb{N}$ is negated by the simple pole of $\Gamma(s)$ when $s = -k$.

We apply Theorem 5 to $\xi'(s)/\xi(s)$ to prove Proposition 1.

Proposition 1. *There exists some constant $B > 0$ such that for all $s \neq 1$,*

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s} - \frac{1}{s-1} + \frac{\log\pi}{2} - \frac{\Gamma'(s/2)}{2\Gamma(s/2)} + \sum_{\text{zeroes}\,\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Proof. By Theorem 5, there exist constants $A, B > 0$ such that

$$\xi(s) = e^{A+Bs}\prod_{\text{zeroes}\,\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho},$$

where the product is over the zeroes of $\xi(s)$, i.e., the nontrivial zeroes of $\zeta(s)$. Taking the logarithmic derivative implies that

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\text{zeroes}\,\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

and the definition of $\xi(s)$ also gives

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{\log\pi}{2} + \frac{\Gamma'(s/2)}{2\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}.$$

This implies that

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s} - \frac{1}{s-1} + \frac{\log\pi}{2} - \frac{\Gamma'(s/2)}{2\Gamma(s/2)} + \sum_{\text{zeroes}\,\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

as required. $\square$

2.3. **Proof of the Classical Zero-Free Region.** We use the Riemann-von Mangoldt formula to prove Theorem 1.

Theorem 1. *There exists an absolute constant $C > 0$ such that the region*

$$\left\{ s \in \mathbb{C} \;\middle|\; \Re(s) > 1 - \frac{C}{\log(\Im(s))} \text{ and } \Im(s) \geq 2 \right\}$$

contains no zeroes of $\zeta(s)$.

First, we present the main ideas to motivate the proof. The Riemann-von Mangoldt formula from Proposition 1 is fundamental as it gives a relationship between $\zeta'(s)/\zeta(s)$ and the nontrivial zeroes of $\zeta(s)$. Proposition 3 gives another identity for $\zeta'(s)/\zeta(s)$. To prove the theorem, we analyze the real parts of both relations to obtain a bound for the real parts of the zeroes. In particular, we employ a clever trigonometric identity to obtain a nonnegativity bound on a linear combination of $\Re(\zeta'(s)/\zeta(s))$ at $s = \sigma$, $s = \sigma + it$, and $s = \sigma + 2it$, which removes the oscillation given by the cosine terms obtained by taking the real part. Combining these bounds implies the theorem.

Proof. By Proposition 3, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} \log p \sum_{r=1}^{\infty} p^{-rs} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Taking the real part gives

$$\Re\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = \Re\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\Re(s)}} \cos(\Im(s)\log n) \tag{10}$$

We now use *Merten's equality* [16] that $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0$, which holds for all $\theta \in \mathbb{R}$. In particular, $3 + 4\cos(\Im(s)\log n) + \cos(2\Im(s)\log n) \geq 0$, so the appropriate linear combination of $\Re(-\zeta'(s)/\zeta(s))$ for $s = \sigma$, $s = \sigma + it$, and $s = \sigma + 2it$ using (10) gives

$$\begin{aligned} 3\left(-\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) + 4\Re\left(-\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) + \Re\left(-\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right) &= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}\big(3 + 4\cos(t\log n) \\ &\qquad + \cos(2t\log n)\big) \\ &\geq 0. \end{aligned} \tag{11}$$

We now analyze the same linear combination using the Riemann-von Mangoldt formula. In the vertical strip where $t \geq 2$ and $1 \leq \sigma \leq 2$, $\Gamma'(s/2)/\Gamma(s/2) = O(\log t)$ by

Stirling's formula. In the same region, by Proposition 1, we have

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= \frac{1}{s-1} - B - \frac{\log \pi}{2} + \frac{\Gamma'(s/2)}{2\Gamma(s/2)} - \sum_{\text{zeroes }\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \\
&= O(\log t) - \sum_{\text{zeroes }\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).
\end{aligned} \tag{12}
$$

In particular, taking the real part gives

$$
\Re\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = O(\log t) - \sum_{\text{zeroes }\rho} \left(\Re\left(\frac{1}{s-\rho}\right) + \Re\left(\frac{1}{\rho}\right)\right) \tag{13}
$$

The sum over the zeroes ρ is positive since if we take $\rho = \beta + i\gamma$, then $\Re(1/\rho) = \beta/|\rho|^2$ and $\Re(1/(s-\rho)) = (\sigma - \beta)/|s-\rho|^2$. In particular, this with (13) implies

$$
-\Re\left(\frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + it)}\right) \leq O(\log t). \tag{14}
$$

For the $s = \sigma + it$ term, take some zero $\rho = \beta + it$ and bound

$$
-\Re\left(\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right) \leq O(\log t) - \Re\left(\frac{1}{s-\rho}\right) = O(\log t) - \frac{1}{\sigma - \beta}. \tag{15}
$$

Finally, since $\zeta(s)$ exhibits a simple pole at $s = 1$, we have

$$
-\frac{\zeta'(\sigma)}{\zeta(\sigma)} \leq \frac{1}{\sigma - 1} + O(1). \tag{16}
$$

Combining (14), (15), and (16) with the nonnegativity linear combination (13) implies that there exists some constant $C > 0$ such that

$$
\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + C \log t \geq 0.
$$

Let $\delta = 1 - \beta$, the distance from ρ to the real line $\Re(s) = 1$, and consider $\sigma = 1 + 4\delta$, which satisfies $1 \leq \sigma \leq 2$. Then the above condition becomes

$$
0 \leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + C \log t = \frac{3}{4\delta} - \frac{4}{5\delta} + C \log t = -\frac{1}{20\delta} + C \log t,
$$

which immediately implies $1/20\delta \leq C \log t$. Thus the real component of the location of the zero ρ is given by $\beta < 1 - C/\log t$, as required. $\square$

3. Proof of the Prime Number Theorem

In this section, we prove the improvement of the prime number theorem presented in Theorem 2. The proof involves transferring bounds between intermediary functions. We first state the following theorem that expresses a partial sum of some Dirichlet series in terms of an integral.

Theorem 6. *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n/n^s$ be a Dirichlet series that is absolutely convergent on the half plane $\{s \in \mathbb{C} \mid \Re(s) > \sigma_a\}$. If $\sigma > \max(0, \sigma_a)$ and $x > 0$, then*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \alpha(s) \frac{x^s}{s}\, ds + R,$$

where the remainder term R is bounded by

$$R = O\left(\sum_{\substack{x/2<n<2x \\ n \neq x}} |a_n| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{4^\sigma + x^\sigma}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^s} \right).$$

We refer the reader to Davenport [6] and Montgomery and Vaughan [17] for details. The proof of Theorem 6 relies on Perron's formula.

3.1. **Upper Bound on the Chebyshev Function.** We begin with the Chebyshev function $\psi(x)$.

Definition 3. For all $x \geq 0$, the *Chebyshev function* $\psi(x)$ is given by

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

We prove that $\psi(x)$ satisfies the following asymptotic relation.

Lemma 4. *For $x \geq 2$, there exists an absolute constant $C > 0$ such that*

$$\psi(x) = x + O\left(x \exp\left(-C\sqrt{\log x}\right)\right).$$

Proof. Let $1 < \sigma < 2$ to be chosen later. Proposition 3 and Theorem 6 give

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s}\, ds + R = -\frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}\, ds + R, \tag{17}$$

where by Proposition 3 the remainder term R is bounded by

$$\begin{aligned} R &= O\left(\sum_{\substack{x/2<n<2x \\ n \neq x}} \Lambda(n) \min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^\sigma}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \right) \\ &= O\left(\sum_{\substack{x/2<n<2x \\ n \neq x}} \Lambda(n) \min\left(1, \frac{x}{T|x-n|}\right) - \frac{(4x)^\sigma}{T} \frac{\zeta'(\sigma)}{\zeta(\sigma)} \right) \end{aligned}$$

We first bound R. Note that $1 \le (x/T) \cdot 1/|x-n|$ for only a constant number of ns, so $1 > (x/T) \cdot 1/|x-n|$ for $O(x)$ many ns. Since $\Lambda(n) \le \log n = O(\log x)$ we have

$$\begin{aligned}\sum_{\substack{x/2<n<2x\\ n\neq x}} \Lambda(n) \min\left(1, \frac{x}{T|x-n|}\right) &= O\left((\log x)\left(1 + \frac{x}{T}\sum_{1\le k\le x}\frac{1}{k}\right)\right)\\ &= O\left(\log x + \frac{x}{T}(\log x)^2\right),\end{aligned}$$

whereas $\zeta'(\sigma)/\zeta(\sigma) = 1/(\sigma-1) + O(1)$ from (16). Now we choose $\sigma = 1 + 1/\log x$ so that the bound on R becomes

$$R = O\left(\log x + \frac{x}{T}(\log x)^2 + \frac{(4x)^\sigma}{T(\sigma-1)}\right) = O\left(\frac{x}{T}(\log x)^2\right). \tag{18}$$

We now bound the integral in (17). Let $\beta = 1 - C/\log T$ for some $0 < C < 1$, and let γ be a piecewise linear curve from $\sigma - iT$ to $\sigma + iT$ to $\beta + iT$ to $\beta - iT$ to $\sigma - iT$. By Theorem 1, $\zeta'(s)/\zeta(s)$ is analytic everywhere except for a simple pole with residue -1 at $s = 1$ (since $\zeta(s) \neq 0$ in the region bounded by γ by the classical zero-free region), so Cauchy's residue theorem implies that

$$-\frac{1}{2\pi i}\oint_\gamma \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds = x.$$

On the other hand,

$$-\frac{1}{2\pi i}\oint_\gamma \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds = -\frac{1}{2\pi i}\left(\int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{\beta+iT} + \int_{\beta+iT}^{\beta-iT} + \int_{\beta-iT}^{\sigma-iT}\right)\left(\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds\right).$$

From (12) we have the bounds

$$\left|-\frac{1}{2\pi i}\int_{\sigma+iT}^{\beta+iT}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds\right| = O\left(\frac{\log T}{T}x^\sigma(\sigma-\beta)\right) = O\left(\frac{x}{T}\right),$$

and using the same reasoning we also obtain

$$\left|-\frac{1}{2\pi i}\int_{\beta-iT}^{\sigma-iT}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds\right| = O\left(\frac{\log T}{T}x^\sigma(\sigma-\beta)\right) = O\left(\frac{x}{T}\right),$$

Finally, we bound the last integral by

$$\left|-\frac{1}{2\pi i}\int_{\beta+iT}^{\beta-iT}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds\right| \le x^\beta\int_T^{-T}\left|\frac{\zeta'(\beta+it)}{\zeta(\beta+it)}\right|\frac{dt}{|\beta+it|} = O\left(x^\beta(\log T)^2\right)$$

Therefore, since $\beta = 1 - C/\log T$ we obtain

$$\psi(x) = x + O\left(\frac{x}{T}(\log x)^2 + x^\beta(\log x)^2\right) = x + O\left(x(\log x)^2\left(\frac{1}{T} + x^{-C/\log T}\right)\right),$$

and choosing $T = \exp\left(\sqrt{C\log x}\right)$ gives the desired error term. $\square$

3.2. **The Prime Number Theorem.** We now use Lemma 4 to obtain the desired asymptotic for $\pi(x)$. We first transition to an intermediary function $\Pi(x)$.

Definition 4. For all $x \geq 0$, $\Pi(x)$ is given by

$$\Pi(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n},$$

We prove the following bound.

Lemma 5. *For $x \geq 2$, there exists an absolute constant $C > 0$ such that*

$$\Pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{\log x}\right)\right).$$

Proof. First note $\Pi(x)$ is expressible in terms of $\psi(x)$ as

$$\Pi(x) = \int_2^x \frac{d\psi(t)}{\log t} = \int_2^x \frac{\psi(t)}{t \log^2 t}\, dt + \frac{\psi(x)}{\log x} - \frac{\psi(2)}{\log 2} = \int_2^x \frac{\psi(t)}{t \log^2 t}\, dt + \frac{\psi(x)}{\log x} + O(1).$$

We now use the bound given by Lemma 4. This implies that

$$\Pi(x) = \int_2^x \frac{t\, dt}{t \log^2 t} + \frac{x}{\log x} + O\left(\int_2^x \frac{\exp\left(-C\sqrt{\log t}\right)}{\log^2 t}\, dt\right) + O\left(\frac{x \exp\left(-C\sqrt{\log x}\right)}{\log x}\right)$$

for some constant $C > 0$, where by integration by parts the first two terms reduce to

$$\int_2^x \frac{t\, dt}{t \log^2 t} + \frac{x}{\log x} = \int_2^x t \frac{d}{dt}\left(-\frac{1}{\log t}\right) dt + \frac{x}{\log x} = \int_2^x \frac{dt}{\log t} + \frac{2}{\log 2} = \int_2^x \frac{dt}{\log t} + O(1).$$

Combining these two equations we obtain

$$\begin{aligned}
\Pi(x) &= \int_2^x \frac{dt}{\log t} + O\left(\int_2^x \frac{\exp\left(-C\sqrt{\log t}\right)}{\log^2 t}\, dt\right) + O\left(\frac{x \exp\left(-C\sqrt{\log x}\right)}{\log x}\right) \\
&= \int_2^x \frac{dt}{\log t} + O\left(\int_2^x \exp\left(-C\sqrt{\log t}\right) dt\right) + O\left(x \exp\left(-C\sqrt{\log x}\right)\right).
\end{aligned} \tag{19}$$

It thus remains to consider the estimate on the error term. Note that

$$\int_2^{x^{1/4}} \exp\left(-C\sqrt{\log t}\right) dt \leq \int_2^{x^{1/4}} dt \leq x^{1/4}, \tag{20}$$

whereas in the range $t \geq x^{1/4}$ we have $\sqrt{\log t} \geq \sqrt{\log x}/2$, so

$$\int_{x^{1/4}}^x \exp\left(-C\sqrt{\log t}\right) dt \leq \int_{x^{1/4}}^x \exp\left(-C\sqrt{\log x}/2\right) dt \leq O\left(x \exp\left(-C'\sqrt{\log x}\right)\right), \tag{21}$$

where $C' = C/2$. Combining the bounds (20) and (21) with (19) gives the result. □

We now prove Theorem 2 by expressing $\pi(x)$ in terms of $\Pi(x)$.

Theorem 2. *For $x \geq 2$, there exists an absolute constant $C > 0$ such that*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{\log x}\right)\right).$$

Proof. By definition of $\Pi(x)$, we have

$$\Pi(x)=\sum_{n\le x}\frac{\Lambda(n)}{\log n}=\sum_{k=1}^{\infty}\sum_{\substack{p\,\text{prime}\\ p^k\le x}}\frac{\Lambda\left(p^k\right)}{\log p^k}=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\substack{p\,\text{prime}\\ p^k\le x}}1=\sum_{k=1}^{\infty}\frac{\pi\left(x^{1/k}\right)}{k}.$$

Therefore, since $\pi\left(x^{1/k}\right)\le x^{1/k}$ for all integers $k\ge 2$, we obtain

$$|\pi(x)-\Pi(x)|=\sum_{k=2}^{\infty}\frac{\pi\left(x^{1/k}\right)}{k}\le\sum_{k=2}^{\infty}x^{1/k}=O\left(x^{1/2}\right).$$

Using Lemma 5, we obtain

$$\pi(x)=\Pi(x)+O\left(x^{1/2}\right)=\int_2^x\frac{dt}{\log t}+O\left(x\exp\left(-C\sqrt{\log x}\right)\right)$$

for some constant $C>0$, as required. $\square$

4. The Classical Zero-Free Region of $\zeta_B(s)$

In this section, we give a proof sketch of the zero-free region of the Beurling zeta function $\zeta_B(s)$ corresponding to a Beurling generalized number system.

Definition 5. A *Beurling generalized number system* B consists of a pair $(\mathscr{P},\mathscr{N})$, where $\mathscr{P}$ is the set of *generalized primes* and $\mathscr{N}$ is the set of *generalized integers*. Here, $\mathscr{P}=\{\lambda_j\}_{j\in\mathbb{N}}$ consists of increasing real numbers $1<\lambda_1\le\lambda_2\le\cdots$ such that $\lambda_j\to\infty$, and $\mathscr{N}$ consists of all finite products generated by $\mathscr{P}$, i.e.,

$$\mathscr{N}=\left\{n\in\mathbb{R}\;\middle|\;n=\lambda_1^{k_1}\cdots\lambda_r^{k_r}\text{ for }\lambda_1,\ldots,\lambda_r\in\mathscr{P}\text{ and }k_1,\ldots,k_r\in\mathbb{N}\right\}.$$

Definition 6. The *Beurling zeta function* ζ_B corresponding to a Beurling generalized number system B is given for $\Re(s)>1$ by

$$\zeta_B(s)=\sum_{n\in\mathscr{N}}\frac{1}{n^s}.$$

Definition 7. Given a Beurling generalized number system B, define $N_B(x)$ to be the function that counts (with multiplicity) the number of generalized integers in $\mathscr{N}$ less than x.

4.1. Beurling-type Measures and the Riemann-von Mangoldt Formula.

In the sequel, we fix a Beurling generalized number system $B=(\mathscr{P},\mathscr{N})$ and assume that $N_B(x)=\kappa x+O\left(x^\theta\right)$ for some $\kappa>0$ and $0\le\theta<1$. We now define the discrete measure $\mathbf{N}_B$ by

$$\mathbf{N}_B=\sum_{n\in\mathscr{N}}\delta_{\log n}.$$

Then using the same application of the monotone convergence theorem and Fubini's theorem, we may switch the summation and integral to obtain

$$\mathcal{L}\{\mathbf{N}_B\}(s)=\int_0^\infty e^{-st}\,d\mathbf{N}_B=\int_0^\infty\sum_{n\in\mathscr{N}}e^{-st}\,d\delta_{\log n}(t)=\sum_{n\in\mathscr{N}}\int_0^\infty e^{-st}\,d\delta_{\log n}(t)=\zeta_B(s)$$

when $\Re(s) > 1$. It follows that $\zeta_B(s)$ has a simple pole at $s = 1$.

Along the same lines as Lemma 1, we give a meromorphic extension of $\zeta(s)$ to the region $\{s \in \mathbb{C} \mid \Re(s) > \theta\}$.

Lemma 6. *For $\Re(s) > \theta$,*

$$\zeta_B(s) = \frac{\kappa s}{s-1} + G(s)$$

for some function $G(s)$ analytic in the region $\{s \in \mathbb{C} \mid \Re(s) > \theta\}$.

Proof. Recall that $N_B(x) = \kappa x + E_B(x)$, where $E_B(x) = O\left(x^\theta\right)$. A similar reasoning gives

$$\begin{aligned}\zeta_B(s) &= s\int_1^\infty N_B(x)x^{-s-1}\,dx \\ &= s\int_1^\infty (\kappa x + E_B(x))x^{-s-1}\,dx \\ &= \frac{\kappa s}{s-1} + s\int_1^\infty E_B(x)x^{-s-1}\,dx,\end{aligned}$$

and since $|E_B(x)x^{-s-1}| = O\left(x^{\theta-\Re(s)-1}\right)$, the last integral converges for $\Re(s) > \theta$. □

For comparison, in the proof of Lemma 1 where B is the pair of the standard prime numbers and natural numbers $\mathbb{N}$, we have $N_B(x) = [x]$ and $E(x) = O(1)$, so $\theta = 0$.

Definition 8. The *Beurling von Mangoldt function* $\Lambda_B(n)$ associated with B is

$$\Lambda_B(n) = \begin{cases} \log\lambda & \text{if } n = \lambda^k \text{ for some integer } k \geq 1 \text{ and some } \lambda \in \mathscr{P}, \\ 0 & \text{otherwise.}\end{cases}$$

We have an analogous version of Proposition 3. We omit the proof as it is similar.

Proposition 4. *For $\Re(s) > 1$,*

$$-\frac{\zeta_B'(s)}{\zeta_B(s)} = \sum_{n\in\mathscr{N}} \frac{\Lambda_B(n)}{n^s}.$$

Finally, we have an analogous Riemann-von Mangoldt formula (cf. Proposition 1).

Proposition 5. *For all $s \neq 1$,*

$$\frac{\zeta_B'(s)}{\zeta_B(s)} = -\frac{1}{s-1} + \sum_{\substack{\textit{zeroes}\,\rho \\ |\rho-s|<1}} \frac{1}{s-\rho} + O\left(\frac{\log\Im(s)}{1-\theta}\right).$$

Unlike the proof of Proposition 1, there is no functional equation for $\zeta_B(s)$. Instead, the proof of Proposition 5 relies on local analytic lemmas such as Jensen's inequality and the Borel-Carathéodory lemma. This approach was first used by Landau in [11], [12], and [13]. We omit the proof as it is technical.

4.2. **Proof Sketch of the Zero-Free Region.** We now give a proof sketch of Theorem 3.

Theorem 3. *Suppose that* $N_B(x) = \kappa x + O\left(x^\theta\right)$ *for some* $\kappa > 0$ *and* $0 \le \theta < 1$. *Then there exists an absolute constant* $C > 0$ *such that the region*

$$\left\{ s \in \mathbb{C} \;\middle|\; \Re(s) > 1 - \frac{C(1-\theta)}{\log(\Im(s))} \text{ and } \Im(s) \ge 2 \right\}$$

contains no zeroes of $\zeta_B(s)$.

Proof. Proposition 4 implies that

$$\Re\left(-\frac{\zeta_B'(s)}{\zeta_B(s)}\right) = \Re\left(\sum_{n\in\mathcal{N}} \frac{\Lambda_B(n)}{n^s}\right) = \sum_{n\in\mathcal{N}} \frac{\Lambda_B(n)}{n^s}\cos(\Im(s)\log n).$$

By Merten's inequality [16], $3 + 4\cos(\Im(s)\log n) + \cos(2\Im(s)\log n) \ge 0$, so

$$3\left(-\frac{\zeta_B'(\sigma)}{\zeta_B(\sigma)}\right) + 4\Re\left(-\frac{\zeta_B'(\sigma+it)}{\zeta_B(\sigma+it)}\right) + \Re\left(-\frac{\zeta_B'(\sigma+2it)}{\zeta_B(\sigma+2it)}\right) \ge 0. \tag{22}$$

We now analyze the same linear combination using Proposition 5 in the vertical strip where $t \ge 2$ and $1 \le \sigma \le 2$. Proposition 5 implies

$$\Re\left(-\frac{\zeta_B'(s)}{\zeta_B(s)}\right) = \Re\left(\frac{1}{s-1}\right) - \sum_{\substack{\text{zeroes }\rho\\ |\rho - s|<1}} \Re\left(\frac{1}{s-\rho}\right) + O\left(\frac{\log\Im(s)}{1-\theta}\right).$$

The sum over the zeroes ρ is again positive. This means that

$$-\Re\left(\frac{\zeta_B'(\sigma+2it)}{\zeta_B(\sigma+2it)}\right) \le O\left(\frac{\log t}{1-\theta}\right). \tag{23}$$

For the $s = \sigma + it$ term, take some zero $\rho = \beta + it$. Since $|s-\rho| < 1$, we bound

$$-\Re\left(\frac{\zeta_B'(\sigma+it)}{\zeta_B(\sigma+it)}\right) \le O\left(\frac{\log t}{1-\theta}\right) - \Re\left(\frac{1}{s-\rho}\right) = O\left(\frac{\log t}{1-\theta}\right) - \frac{1}{\sigma-\beta}. \tag{24}$$

Finally, since $\zeta_B(s)$ exhibits a simple pole at $s = 1$, we have

$$-\frac{\zeta_B'(\sigma)}{\zeta_B(\sigma)} \le \frac{1}{\sigma-1} + O(1). \tag{25}$$

Combining (23), (24), and (25) with the nonnegativity linear combination (22) implies that there exists some constant $C > 0$ such that

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + \frac{\log t}{C(1-\theta)} \ge 0.$$

Let $\delta = 1-\beta$, the distance from ρ to the real line $\Re(s) = 1$, and consider $\sigma = 1 + 4\delta$, which satisfies $1 \le \sigma \le 2$. Then the above condition becomes

$$0 \le \frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + \frac{\log t}{C(1-\theta)} = \frac{3}{4\delta} - \frac{4}{5\delta} + \frac{\log t}{C(1-\theta)} = -\frac{1}{20\delta} + \frac{\log t}{C(1-\theta)},$$

which immediately implies $1/20\delta \leq \log t/C(1-\theta)$. Thus, the real component of the location of the zero ρ is given by $\beta < 1 - C(1-\theta)/\log t$, as required. $\square$

5. Proof of the Beurling Prime Number Theorem

Here, we sketch a proof of the generalized Beurling prime number theorem. Then we give a discussion of the optimality of these bounds as shown by Diamond, Montgomery, and Vorhauer [5].

5.1. **Proof Sketch.** We begin with the analogue of the Chebyshev function and its asymptotics.

Definition 9. For all $x \geq 0$, the *Beurling Chebyshev function* $\psi_B(x)$ associated with B is given by

$$\psi_B(x) = \sum_{\substack{n \in \mathcal{N} \\ n \leq x}} \Lambda_B(n).$$

Lemma 7. *For $x \geq 2$, there exists an absolute constant $C > 0$ such that*

$$\psi_B(x) = x + O\left(x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right).$$

Proof. Along the same lines as in Lemma 4, we choose $\sigma = 1 + 1/\log x > 1$ so that by Proposition 4 we have

$$\psi_B(x) = \sum_{\substack{n \in \mathcal{N} \\ n \leq x}} \Lambda_B(n) = \frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma + iT} \sum_{n \in \mathcal{N}} \frac{\Lambda_B(n)}{n^s}\frac{x^s}{s}\, ds + R = -\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma + iT} \frac{\zeta_B'(s)}{\zeta_B(s)}\frac{x^s}{s}\, ds + R$$

where $R = O\left(x(\log x)^2/T\right)$.

For the integral, let $\beta = 1 - C(1-\theta)/\log T$ for some $0 < C < 1$, and let γ be a piecewise linear curve from $\sigma - iT$ to $\sigma + iT$ to $\beta + iT$ to $\beta - iT$ to $\sigma - iT$. But by Theorem 3, $\zeta_B'(s)/\zeta_B(s)$ is analytic everywhere except for a simple pole with residue -1 at $s = 1$ (because $\zeta_B(s) \neq 0$ in the region described by Theorem 4), so Cauchy's residue theorem implies that

$$-\frac{1}{2\pi i}\oint_\gamma \frac{\zeta_B'(s)}{\zeta_B(s)}\frac{x^s}{s}\, ds = x.$$

On the other hand,

$$-\frac{1}{2\pi i}\oint_\gamma \frac{\zeta_B'(s)}{\zeta_B(s)}\frac{x^s}{s}\, ds = -\frac{1}{2\pi i}\left(\int_{\sigma - iT}^{\sigma + iT} + \int_{\sigma + iT}^{\beta + iT} + \int_{\beta + iT}^{\beta - iT} + \int_{\beta - iT}^{\sigma - iT}\right)\left(\frac{\zeta_B'(s)}{\zeta_B(s)}\frac{x^s}{s}\, ds\right).$$

Using $|\zeta_B'(s)/\zeta_B(s)| = O(\log T/(1-\theta))$ in this region and $\beta = 1 - C(1-\theta)/\log T$, we obtain that the integral contributes

$$x + O\left(x^\beta (\log x)^2\right).$$

Therefore, the total contribution is

$$\psi_B(x) = x + O\left(\frac{x}{T}(\log x)^2 + x^\beta(\log x)^2\right) = x + O\left(x(\log x)^2\left(\frac{1}{T} + x^{-C(1-\theta)/\log T}\right)\right),$$

where choosing $T = \exp\left(\sqrt{(1-\theta)\log x}\right)$ gives the desired bound. $\square$

We translate the bound on $\psi_B(x)$ to a bound on $\Pi_B(x)$.

Definition 10. For all $x \geq 0$, $\Pi_B(x)$ is given by

$$\Pi_B(x) = \sum_{\substack{n \in \mathscr{N} \\ n \leq x}} \frac{\Lambda(n)}{\log n}.$$

Lemma 8. *For all $x \geq 2$, there exists an absolute constant $C > 0$ such that*

$$\Pi_B(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right).$$

Proof. The proof follows exactly the same as that of Lemma 5: Write

$$\Pi_B(x) = \int_2^x \frac{d\psi_B(t)}{\log t} = \int_2^x \frac{\psi_B(t)}{t \log^2 t}\, dt + \frac{\psi_B(x)}{\log x} + O(1)$$

and use the bound from Lemma 7 to obtain

$$\begin{aligned}
\Pi_B(x) = &\int_2^x \frac{t\, dt}{t \log^2 t}\, dt + \frac{x}{\log x} \\
&+ O\left(\int_2^x \frac{\exp\left(-C\sqrt{(1-\theta)\log t}\right)}{\log^2 t}\, dt + \frac{x \exp\left(-C\sqrt{(1-\theta)\log x}\right)}{\log x}\right) \\
= &\int_2^x \frac{dt}{\log t} \\
&+ O\left(\int_2^x \exp\left(-C\sqrt{(1-\theta)\log t}\right)\, dt + x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right) \\
= &\int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right)
\end{aligned}$$

for some constant $C > 0$. $\square$

Finally, we translate the bound on $\Pi(x)$ to obtain a bound on $\pi_B(x)$.

Theorem 4. *Suppose that $N_B(x) = \kappa x + O\left(x^\theta\right)$ for some $\kappa > 0$ and $0 \leq \theta < 1$. Then there exists an absolute constant $C > 0$ such that for all $x \geq 2$,*

$$\pi_B(x) = \int_2^\infty \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right).$$

Proof. By definition of $\Pi_B(x)$, we have

$$\Pi_B(x) = \sum_{\substack{n \in \mathscr{N} \\ n \leq x}} \frac{\Lambda_B(n)}{\log n} = \sum_{k=1}^\infty \sum_{\substack{\lambda \in \mathscr{P} \\ \lambda^k \leq x}} \frac{\Lambda_B\left(\lambda^k\right)}{\log \lambda^k} = \sum_{k=1}^\infty \frac{1}{k} \sum_{\substack{\lambda \in \mathscr{P} \\ \lambda^k \leq x}} 1 = \sum_{k=1}^\infty \frac{\pi_B\left(x^{1/k}\right)}{k}.$$

Therefore, since $\pi_B\left(x^{1/k}\right) \leq x^{1/k}$ for all integers $k \geq 2$, we obtain

$$|\pi_B(x) - \Pi_B(x)| = \sum_{k=2}^{\infty} \frac{\pi_B\left(x^{1/k}\right)}{k} \leq \sum_{k=2}^{\infty} x^{1/k} = O\left(x^{1/2}\right).$$

Using Lemma 8, we obtain

$$\pi_B(x) = \Pi_B(x) + O\left(x^{1/2}\right) = \int_2^x \frac{dt}{\log t} + O\left(x \exp\left(-C\sqrt{(1-\theta)\log x}\right)\right)$$

for some constant $C > 0$, as required. □

5.2. **Optimality.** Remarkably, Diamond, Montgomery, and Vorhauer [5] proved that this bound is optimal in the following sense.

Theorem 7. *If $\theta \in (1/2, 1)$ and $a > (4/e)(1-\theta)$, then there exists some Beurling generalized number system $B = (\mathscr{P}, \mathscr{N})$ such that*

(i) $N_B(x) = \kappa x + O\left(x^{\theta}\right)$ for some $\kappa > 0$;
(ii) $\zeta_B(s)$ is analytic for $\sigma \geq \theta$ with a simple pole at $s = 1$;
(iii) $\zeta_B(s)$ has no zero to the right of the curve $\sigma = 1 - a/\log \Im(s)$ for $\Im(s) \geq 2$;
(iv) $\psi_B(x) - x = \Omega_{\pm}\left(x \exp\left(-2\sqrt{a \log x}\right)\right)$; and
(v) $\psi_B(x) = x + O\left(x \exp\left(-2\sqrt{a \log x}\right)\right)$.

In other words, for any choice of $\theta \in (1/2, 1)$, they constructed a Beurling generalized number system such that $\zeta_B(s)$ and $\psi_B(x)$ satisfies comparable properties to the standard Riemann zeta function $\zeta(s)$ and Chebyshev function $\psi(x)$, such as the classical zero-free region and the prime number theorem error bound.

Their construction of B involves constructing a continuous prime counting function $\Pi_C(x)$ such that its zeta function has the desired properties above, then followed by a probabilistic method that shows the existence of a discrete measure $\pi_B(x)$ that is sufficiently close to $\Pi_C(x)$ so that B has the desired properties.

Define the entire function

$$G(z) = \begin{cases} 1 - \frac{e^{-z} - e^{-2z}}{z} & \text{if } z \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\gamma_k\}_{k \in \mathbb{N}}$ be some sequence of strictly increasing real numbers such that $\gamma_k \to \infty$, with $\gamma_k = -\gamma_{-k}$ for integers $k < 0$. For $k \neq 0$, define $\rho_k = (1 - a/\log|\gamma_k|) + \gamma_k i$ and $\ell_k = \alpha \log|\gamma_k|$ for some $\alpha > 0$. Then for $\Re(s) > 1$ define the continuous zeta function

$$\zeta_C(s) = \frac{s}{s-1} \prod_{k \in \mathbb{Z}\setminus\{0\}} G(\ell_k(s - \rho_k)).$$

From this definition, we deduce that $\zeta_C(s)$ has zeroes at the points ρ_k. This choice of G is primarily motivated by the following. Recall that Lemma 3 implies that

$$\log \zeta(s) = \int_0^{\infty} e^{-st}\, dP(t) = \sum_{p \text{ prime}} \sum_{r=1}^{\infty} \frac{1}{rp^{rs}} = \int_1^{\infty} x^{-s}\, d\Pi(x)$$

for $\Re(s) > 1$. Similarly, to relate $\zeta_C(s)$ to some set of primes with measure $d\Pi_C(v)$, we write for $\Re(s) > 1$ that

$$\log \zeta_C(s) = \int_1^\infty v^{-s}\, d\Pi_C(v). \tag{26}$$

Let χ be the characteristic function of the interval $[e, e^2]$, and let $\star$ denote the multiplicative convolution of f and g, i.e.,

$$(f \star g) = \int_1^u f(v) g\left(\frac{u}{v}\right) \frac{dv}{v}.$$

It follows that

$$\left(\frac{e^{-z} - e^{-2z}}{z}\right)^n = \int_1^\infty \chi^{\star n}(u) u^{-z-1}\, du,$$

where $\chi^{\star n}$ denotes the n-fold convolution. Thus, if we define

$$f(u) \coloneqq \sum_{n=1}^\infty \frac{\chi^{\star n}(u)}{n},$$

then for $\Re(z) > 1$ we have

$$\log G(z) = \log\left(1 - \frac{e^{-z} - e^{-2z}}{z}\right) = -\sum_{n=1}^\infty \frac{1}{n}\left(\frac{e^{-z} - e^{-2z}}{z}\right)^n = -\int_e^\infty f(u) u^{-z-1}\, du.$$

From this and

$$\log \zeta_C(s) = \log\left(\frac{s}{s-1}\right) + \sum_{k \in \mathbb{Z}\setminus\{0\}} \log G(\ell_k(s - \rho_k))$$

one can show that

$$d\Pi_C(v) = \left(\frac{1 - 1/v}{\log v} - 2 \sum_{\substack{k \geq 1 \\ \exp(\ell_k) \leq v}} \frac{f\left(v^{1/\ell_k}\right)}{\ell_k} v^{a/\log|\gamma_k|} \cos(\gamma_k \log v)\right) dv$$

is a continuous measure that satisfies (26), provided that the γ_k are chosen such that they increase sufficiently fast for $d\Pi_C$ to converge. Choosing $\gamma_k = \gamma_1^{A^{k-1}}$ for γ_1 and A large enough suffices.

The authors in [5] used a probabilistic method approach to construct the Beurling primes $\mathscr{P} = \{\lambda_j\}_{j \in \mathbb{N}}$. Given $d\Pi_C$, they constructed a sequence of strictly increasing real numbers $\{v_k\}_{k \in \mathbb{N}}$ with $v_0 = 1$ such that $v_k \to \infty$. For each $k \in \mathbb{N}$, they define v_k to be a generalized prime in $B = (\mathscr{P}, \mathscr{N})$ with probability

$$p_k = \int_{v_{k-1}}^{v_k} d\Pi_C(v).$$

Due to a technical detail, the v_k are chosen such that they increase sufficiently slowly for $p_k \leq 1/2$. This constraint is so that the Kolmogorov exponential tail bound (see

§18.1 of [15] or §6 Lemma 6 of [5]) can be applied to obtain the bound

$$\int_1^x v^{-it}\, d\pi_B(v) = \int_1^x v^{-it}\, d\Pi_C(v) + O\left(\sqrt{x\log(|t|+2)}\right) \tag{27}$$

for $x \geq 1$ and $t \in \mathbb{R}$. Choosing $v_k = \sqrt{\log k}$ suffices. This completes the construction of B. Finally, the $O\left(\sqrt{x\log(|t|+2)}\right)$ difference in (27) is sufficiently small to obtain an analytical continuation of $\log(\zeta_B(s)/\zeta_C(s))$ to the region $\Re(s) > 1/2$ satisfying

$$\log \frac{\zeta_B(s)}{\zeta_C(s)} = O\left(\sqrt{\log(|\Im(s)|+2)}\right).$$

The same techniques used to show the classical zero-free region and the prime number theorem estimate can be used to prove Theorem 7.

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