DEPENDENT RANDOMIZED ROUNDING ON BIPARTITE GRAPHS

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Abstract. In the context of randomized rounding, *dependent rounding* aims to achieve stronger guarantees on the distribution of values of the rounded constraint and objective values than independent rounding by conditioning on some variables when rounding others. This is especially important when the problem involves strict structural constraints that must be satisfied after rounding. Additionally, dependent rounding sometimes aims to ensure that the cost of the rounded solution does not exceed the cost of the fractional solution. We present a dependent rounding technique for the bipartite matching problem in order to preserve the costs and constraints specific to this problem. We then compare this technique to dependent rounding for a general linear program defined over the edges of a bipartite graph.

1. Introduction

Randomized rounding is a fundamental technique in approximating the solution to a linear program by rounding a fractional solution of a relaxation of the program. The *independent rounding technique* developed by Raghavan and Thompson [6] allows the use of Chernoff bounds to develop tight approximations for various problems such as set cover and min-cuts.

Despite the power of this technique, independent rounding is frequently not viable for problems that require constraints to be satisfied by the rounded solution. *Hard constraints* in problems such as the clustering problem [4] or the job-scheduling problem [2] make independent rounding techniques ineffective since they are not necessarily preserved (with probability one) in the rounded solution.

Dependent rounding techniques were developed to gain more tractable tail bounds on the distribution of the rounded objective value. These schemes achieve this goal by imposing stronger guarantees on constraints and the objective function using a restricted distribution over variables which introduces dependency. Dependent rounding techniques frequently employ *negative correlation* to control the behavior of these variables. Furthermore, most dependent schemes ensure *marginal preservation* in the sense that that the probability that a random variable is 1 is equal to the fractional value of the relaxation. These two properties often lead to Chernoff-type concentration bounds around the optimal objective value.

In this paper, we provide a brief discussion on two novel dependent rounding techniques and their correctness, motivation, and applications to bipartite graph matchings. In particular, we analyze the bipartite perfect matching problem and the general bipartite graph rounding problem.

Notation. In the sequel, $G = (U \cup V, E)$ denotes a bipartite graph with left half U and right half V. For any node $v \in U \cup V$, let

$$\delta(v) \coloneqq \{e \in E \mid v \text{ is incident to } e\},$$
$$\Gamma(v) \coloneqq \{u \in U \cup V \mid u \text{ is a neighbor of } v\}$$

denote the set of incident edges and neighboring vertices of v, respectively. Finally, let n and m denote the number of vertices and edges of G, respectively.

We first introduce the bipartite perfect matching problem.

Definition 1.1 (Bipartite perfect matching problem)**.** Let $c \in \mathbb{R}^E_{\geq 0}$ be the cost vector, associating each edge with a cost. The *bipartite perfect matching problem* asks for a subset $S \subseteq E$ of edges of minimal total cost in G such that every vertex in G is connected to exactly one edge in E.

Let $X \in \{0,1\}^E$ be the indicator vector denoting if each edge is taken. Formally, our integer linear program is

$$\begin{aligned} \text{minimize} \quad & c^\top X \\ \text{subject to} \quad & \sum_{e \in \delta(v)} X_e = 1 \quad \forall\, v \in U \cup V, \\ & X_e \geq 0 \qquad\quad\ \forall\, e \in E. \end{aligned}$$

This problem has hard constraints that there must be exactly one chosen edge incident to each vertex. These constraints must be preserved during the rounding step; otherwise, our solution is no longer a perfect matching.

Arora, Frieze, and Kaplan [1] develop a dependent rounding technique on bipartite perfect matchings in which the degree preservation constraints that maintain the perfect matching are satisfied with high probability and with an additive error. The objective function $c^\top X$ is also within an additive error of the fractional solution $c^\top x$. However, it may be the case that there is a hard budget constraint on $c^\top X$ and an additive error is not acceptable. Rohwedder, Rouhani, and Wennmann [7] develop a dependent rounding algorithm to find a perfect matching in bipartite graphs with cost always preserved ($c^\top X \leq c^\top x$) and the degree preservation constraints always satisfied ($\sum_{e \in \delta(v)} X_e = 1$ for all $v \in U \cup V$).

Harris [3] uses a different approach to satisfy hard constraints while also allowing for significant negative-correlation structure. Concretely, the methods in [3] consider the problem of dependent rounding in *general* bipartite graphs without any perfect matching constraints.

Definition 1.2 (Bipartite rounding problem)**.** Suppose each edge e has weight $y_{\mathsf{e}} \in [0,1]$. The *bipartite rounding problem* asks to generate random variables $\{Y_{\mathsf{e}}\}_{\mathsf{e} \in E}$ such that the following properties hold.

(I) (**Preservation of constraints**) With probability 1, $\sum_{\mathsf{e} \in \delta(v)} Y_{\mathsf{e}} \leq 1$ for all $v \in V$. (Note this is over the right half.)

(II) (**Preservation of expectation**) For all $\mathsf{e} \in E$, we have $\mathbb{E}[Y_{\mathsf{e}}] = y_{\mathsf{e}}$.

(III) (**General nonpositive correlation**) For *some* subsets $L \subseteq E$, we have the *nonpositive correlation* property

$$\mathbb{E}\left[\prod_{\mathsf{e} \in L} Y_{\mathsf{e}}\right] \leq \prod_{\mathsf{e} \in L} y_{\mathsf{e}}.$$

(IV) (**Strong local negative correlation**) For *some* pairs of edges $(\mathsf{e}_1, \mathsf{e}_2)$, we have the *strong negative correlation* property

$$\mathbb{E}[Y_{\mathsf{e}_1} Y_{\mathsf{e}_2}] \leq \phi_{\mathsf{e}_1,\mathsf{e}_2} \cdot y_{\mathsf{e}_1} y_{\mathsf{e}_2},$$

where $\phi_{\mathsf{e}_1,\mathsf{e}_2}$ is a constant that may depend on e_1 and e_2.

Depending on the formulation, the subsets and pairs in properties (III) and (IV) vary. We now give some remarks on the properties that a dependent rounding scheme should have .

One major benefit of independent rounding is that we can analyze the joint distribution of the rounding scheme by marginalizing the individual variables separately. While we cannot keep the entirety of this structure in dependent rounding, we would at least like the *marginals* of every individual variable to be well-understood. The dependent rounding schemes we discuss achieve the property of *marginal preservation*: For each edge e, the probability that $X_{\mathsf{e}} = 1$ is equal to the value of x_{e}.

We would additionally like the joint distribution of the random variables $Z_1, Z_2, \ldots, Z_n$ to be relatively tractable; this allows us to analyze the rounding scheme in a meaningful way. Ideally, the joint distribution should have some measure of negative correlation between the variables.

The goal of obtaining negative correlation is often so that we can guarantee tighter tail bounds on the distribution of some objective. Therefore, the dependent rounding scheme should involve variables with tight and well-understood tail bound properties. For example, consider the machine-scheduling problem. Suppose $\mathcal{M}$ is a set of machines and $\mathcal{J}$ is a set of jobs. Each job $j \in \mathcal{J}$ has some weight w_j and processing time $p_j^{(i)}$ on machine $i \in \mathcal{M}$. The problem asks for an assignment of all jobs to machines in order so that the overall weighted completion time is minimized. This problem can naturally be transformed into an integer linear program. After finding a relaxed solution, one can apply dependent rounding with strong negative correlation properties to convert the relaxed solution to an integral solution with strong approximation ratios.

The organization of the paper is as follows. In Section 2, we present a dependent randomized rounding scheme for the bipartite perfect matching problem and analyze its correctness. In Section 3, we give some brief preliminaries on negative-correlated exponential random variables, which serve as a key subroutine in generalized bipartite rounding. In Section 4, we analyze the correctness of the dependent randomized rounding scheme for the bipartite rounding problem and discuss the main differences between the two schemes.

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2. Bipartite Perfect Matching

We first discuss a method by Rohwedder, Rouhani, and Wennmann [7] to apply dependent rounding to find a perfect matching in bipartite graphs, where exactly one chosen edge must be incident to each vertex. These hard constraints must be preserved during the rounding step; otherwise, our solution no longer a perfect matching. Rohwedder et al. [7] discusses the case in which the cost must be minimized, but this algorithm can be used to maximize the cost by taking the negation of all the edges.

Recall that the integer linear program is

$$\begin{aligned} \text{minimize} \quad & c^\top X \\ \text{subject to} \quad & \sum_{e\in\delta(v)} X_e = 1 \quad \forall v \in U \cup V, \\ & X_e \geq 0 \qquad\quad\;\; \forall e \in E, \end{aligned}$$

where $c \in \mathbb{R}^m_{\geq 0}$ is a cost vector. Let $x \in [0,1]^E$ be the solution to the relaxed linear program. Using Markov's inequality, Gandhi et al. [2] developed a dependent randomized rounding algorithm that established a negatively correlation between the values of X so that

$$c^\top X \leq (1+\epsilon)c^\top x, \tag{2.1}$$

where $\epsilon > 0$ is an arbitrarily small constant that determines the success probability. Such a solution is achieved in randomized polynomial time.

There are further guarantees of this algorithm. Let $a_1, \ldots, a_k \in [0,1]^E$ be any predefined vectors such that for each $j \in \{1,\ldots,k\}$ there is some $v \in U \cup V$ with $\operatorname{supp}(a_j) \subseteq \delta(v)$. The dependent rounding solution presented by Gandhi et al. [2] guarantees the properties

(A) (**Degree Preservation**) that for all vertices $v \in A \cup B$,

$$X(\delta(v)) \in \{\lfloor x(\delta(v))\rfloor, \lceil x(\delta(v))\rceil\}; \text{ and}$$

(B) (**Concentration**) that for all $j \in \{1,\ldots,k\}$,

$$|a_j^\top x - a_j^\top X| \leq O\left(\max\left\{\log k, \sqrt{a_j^\top x \cdot \log k}\right\}\right).$$

Property (A) implies that $\sum_{e\in\delta(v)} X_e = 1$ for all $v \in U \cup V$. The algorithm incrementally rounds the values of x. Each step modifies two values of x and the change made maintains their sum. Therefore, the value of $\sum_{e\in\delta(v)} X_e$ does not change for any vertex v and the final solution is still a perfect matching.

Property (B) guarantees that the values of our predefined linear functions $a_j^\top x$ do not change significantly after rounding. The a_j vectors serve as arbitrary linear measurements of the final solution whose concentration the authors want to preserve. This may be necessary in an application of the bipartite perfect matching algorithm to a problem in which local quantities must be concentrated. An example of this is provided at the bottom of this section, when we show how this algorithm can be applied to the Budgeted Santa Claus Problem.

Rohwedder, Rouhani, and Wennmann [7] improved the approximation (2.1) to

$$c^\top X \leq c^\top x$$

by ensuring that the cost achieved with the rounded variables does not exceed the cost of the relaxed linear program.

There are various applications when this constraint may be necessary. For instance, there might be a hard budget constraint on the total cost $c^\top X$. The authors find such an X from the relaxed solution x in a series of steps. First, in the spirit of property (A), we introduce the notion of a degree-preserving edge set.

Definition 2.1. A *degree-preserving edge set* S is any $S \in \{0,1\}^E$ such that for all $v \in A \cup B$,

$$S(\delta(v)) \in \{\lfloor x(\delta(v)) \rfloor, \lceil x(\delta(v)) \rceil\}.$$

where for any vector $A \in \mathbb{R}^E$, $A(\delta(v))$ is the sum of the values of A indexed by $\delta(v)$.

Let P be the convex hull of all degree-preserving edge sets. Rohwedder, Rouhani, and Wennmann [7] proved the followimg important lemma.

Lemma 2.2. *There exists a convex combination $y = \sum_{i=1}^k \lambda_i S_i$ with the constraints $\lambda_i \in [0,1]$, $\sum_{i\in[k]} \lambda_i = 1$, and $S_i \in P \cup \{0,1\}^E$ such that*

- *$c^\top y \leq c^\top x$ (i.e., the cost does not increase);*
- *$a_j^\top y = a_j^\top x$ for all $j \in \{1,\dots,k\}$ (i.e., the chosen linear functions evaluate to the same values); and*
- *$|\{\mathsf{e} \in \delta(v) \mid y_\mathsf{e} \notin \{0,1\}\}| \leq 2k$ for all $v \in A \cup B$ (i.e., there are at most $2k$ vertices fractionally incident to each vertex with y).*

Step 1 We round y (which exists by Lemma 2.2) to a more discrete assignment y'. In polynomial time, we can compute the assignment $y' = \sum_{i=1}^k \lambda'_i S_i$ where $\sum_{i=1}^k \lambda'_i = 1$ and
- $c^\top y' \leq c^\top y$ (i.e., cost does not increase);
- $\lambda'_i = h_i/2^\ell$ for some $h_i \in \mathbb{Z}_{\geq 0}$;
- $\lambda'_i = \lambda_i$ for all $i \in \{1,\dots k\}$ and flor all $\lambda_i \in \{0,1\}$; and
- $|y_\mathsf{e} - y'_\mathsf{e}| \leq k/2^\ell$ for all edges $\mathsf{e} \in E$.

Note that y' preserves the degree of y since $y' \in P$. This last claim follows from the fact that y' is a convex combination of degree preserving sets of which P is a convex hull.

Step 2 We construct a binary tree $\mathcal{T}$ with levels $0,1,\dots,l$. Each node corresponds to an edge set.
1. First, we set the 2^l nodes at the lowest level l. In the above assignment $y' = \sum_{i\in[k]} \lambda'_i S_i$, we have that $\sum h_i = 2^l$, so for all $i \in [k]$ we set h_i nodes in the lowest level to the assignment S_i. This implies that y' is the average of the edge sets in the lowest level.
2. We then iteratively calculate each level above this by merging the edge sets of each node's two children. Each level $j \in \{1,\dots,l-1\}$ represents a fractional edge set y'_j that is the average of the integral edge sets in the nodes at level j. When constructing each level j, we aim to ensure cost preservation that $c^\top y'_j \leq c^\top y'_{j+1}$.

Our merging algorithm is as follows.

Merge Suppose we are merging two integral edge sets $S_{i,1}$ and $S_{i,2}$ that represent the respective left and right children of a node i at level j. We construct two potential edge sets $T_i, T'_i \in \{0,1\}^E$.
1. We decompose the symmetric difference of $S_{i,1}$ and $S_{i,2}$ into cycles and paths.
2. For each cycle and path, we randomly choose one alternating edge set and place these in T_i, placing the other one in T'_i. This implies that $T_i + T'_i = S_{i,1} + S_{i,2}$. By construction of T_i and T'_i, the degree is preserved; that is,

$$T_i(\delta(v)), T'_i(\delta(v)) \in \left\{ \left\lfloor \frac{S_{i,1}(\delta(v)) + S_{i,2}(\delta(v))}{2} \right\rfloor, \left\lceil \frac{S_{i,1}(\delta(v)) + S_{i,2}(\delta(v))}{2} \right\rceil \right\}$$

for all $v \in U \cup V$. Therefore, the load on each vertex does not exceed 1.

3. Repeat for every node at level j. Construct one fractional assignment from the T_i edge sets and one for the T_i' edge sets by
$$z_j = \frac{1}{2^j} \sum_{i \in [2^j]} T_i \quad \text{and} \quad z_j' = \frac{1}{2^j} \sum_{i \in [2^j]} T_i'.$$
4. We pick the fractional matching out of z_j and z_j' that is lower cost and set the nodes accordingly to either T_i and T_i', and let y_j be the value of z_j or z_j' that yields the lower cost.
Since $T_i + T_i' = S_{i,1} + S_{i,2}$, $(z_j + z_j')/2 = y_{j+1}'$ implies that $(c^\top z_j + c^\top z_j')/2 = c^\top y_{j+1}'$. It follows that $\min(c^\top z_j, c^\top z_j') \leq c^\top y_{j+1}'$ implies that $c^\top y_j' \leq c^\top y_{j+1}'$.

Therefore, the cost of the averaged fractional assignment does not increase as we move to higher levels. We end by setting our integral solution X to the root node's edge set. By our cost-preservation logic from one row to the next, we maintain the invariant that $c^\top X \leq c^\top y'$. Hence, in randomized polynomial time, we can find an integral rounded solution to the bipartite perfect matching program that satisfies degree preservation to maintain the validity of the matching, and satisfies cost preservation so that the cost does not increase with rounding.

Rohwedder, Rouhani, and Wennmann [7] discussed how this algorithm would be applied to the *Budgeted Santa Claus Problem*. This problem consists of a set of n resources R to be distributed among a set of m players P. Each resource j has a value $v_j \geq 0$ and assigning resource j to player i has cost $c_{ij} \geq 0$. There is a total budget $C \geq 0$ that cannot be exceeded. The goal is to split R into disjoint sets R_i where player i receives the resources in set R_i, so that we maximize $min_{i\in P} \sum_{j \in R_i} v_j$ (maximize the value received by the player that receives the least value). We must maintain the total budget constraint $\sum_{i\in P} \sum_{j\in R_i} c_{ij} \leq C$.

The detailed presented by Rohwedder, Rouhani, and Wennmann [7] to solve this problem with bipartite perfect matching is beyond the scope of this paper, but we present an intuitive explanation here.

The problem is relaxed to a fractional assignment in which "big" resources may be assigned fractionally via variables x_{ib} and "small" resources via variables z_{is}, subject to the constraints that each player receives either one big resource or at least T value in small ones, and that each resource is assigned to at most one player. This fractional solution is embedded into a bipartite graph by introducing a dummy node d and connecting each player i to either the big resources or to d with total weight $y_i = 1 - \sum_b x_{ib}$. To control the allocation of small resources during rounding, the authors define one linear function $a^{(s)}$ for each small resource s, supported only on edges incident to d, with coefficients $a^{(s)}_{(i,d)} = z_{is}/y_i$. These functions satisfy $\sum_e y_e a^{(s)}_e \leq 1$, allowing their cost-preserving dependent rounding scheme to maintain all degree constraints while ensuring that every $a(s)^\top X$ remains tightly concentrated. As a consequence, each small resource is selected only $O(\log n)$ times with high probability, yielding a feasible near-optimal integral allocation.

The dependent rounding method discussed in this section rounds solutions to the bipartite perfect matching problem, however; it only maintains the constraints specific to this problem.

3. Negatively-Correlated Exponential Random Variables

Having analyzed bipartite perfect matchings, we turn to the more general problem of dependent randomized rounding in arbitrary bipartite graphs. As mentioned in Section 1, a desired property of this rounding is *strong negative correlation* in the variables (which is infeasible in independent rounding).

To see how negative correlation can be achieved, we present the seminal CorrelatedExponential algorithm in [3]. This algorithm is not directly related to the bipartite rounding problem but serves as the key subroutine call. Hence, a clear explication of the algorithm's results helps us contextualize the bipartite rounding algorithm while also shedding insight into the relation between the bipartite perfect matching problem.

Concretely, the algorithm takes a d-dimensional *rate vector* $\rho \in [0,1]^d$ with $\sum_{i=1}^d \rho_i \leq 1$ as input. The algorithm proceeds as follows.

(Note: A geometric random variable with probability parameter λ refers to the random variable that denotes the number of trials strictly *before* we see the first success of an experiment with success probability λ.)

The CorrelatedExponential algorithm has the following guarantees.

Algorithm 1 CorrelatedExponential

procedure CorrelatedExponential(ρ) ▷ where $\rho \in [0,1]^d$ and $\sum_{i=1}^{d} \rho_i \leq 1$.
  Initialize a d-dimensional vector Z to hold our negatively correlated variables.
  Draw $X \leftarrow$ MultivariateGeometric(ρ).
  for each coordinate $i = 1, \ldots, d$ **do**
    if $\rho_i \in \{0, 1\}$ **then**
      Set $Z_i \sim \mathrm{Exp}(1)$ as an exponential random variable with rate 1.
    else
      Set $\alpha_i = -\log(1 - \rho_i)$.
      Draw a random variable $S_i \in [0,1]$ with density $\alpha_i e^{-\alpha_i s}/\rho_i$.
      Set $Z_i \coloneqq \alpha_i(X_i + S_i)$.
  return Z

(i) Each random variable Z_i satisfies $Z_i \sim \mathrm{Exp}(1)$ (cf. Proposition 3.1).
(ii) The variables $Z_1, \ldots, Z_d$ are negatively correlated (cf. Proposition 3.2).
(iii) The variables $Z_1, \ldots, Z_d$ are negatively associated (cf. Proposition 3.4).

Before we prove these properties, we first note that MultivariteGeometric(ρ) is a direct generalization of a traditional geometric distribution. In particular, the variables $X_1, \ldots, X_d$ are generated by a *single global procedure* in which the ℓth experiment succeeds with probability ρ_ℓ; X_ℓ is the number of total trials before we see the first successful trial of experiment ℓ. The marginal distribution of X_ℓ is precisely geometric with parameter ρ_ℓ.

The CorrelatedExponential algorithm consists of a global and local phase. The *global* phase draws $X \leftarrow$ MultivariateGeometric(ρ); this sets a baseline for the values of $Z_1, \ldots, Z_d$. The *local* phase then modifies the values of the X_i by drawing the variables S_i independently and then rescaling the sum $X_i + S_i$ to form Z_i.

Global: The central negative-correlation property is actually already present in the global phase of the algorithm. To see this, consider the value of the first coordinate X_1 in the vector $X \leftarrow$ MultivariateGeometric(ρ). When conditioned on $X_2, X_3, \ldots, X_d$ having large values, we are more likely to expect that X_1 has a small value (i.e., if it took a long time to see any of experiments $2, 3, \ldots, d$ succeed, it is probably because there was a success of experiment 1 earlier). Furthermore, the multivariate geometric distribution has many of the nice properties we mentioned in our opening intuition: the individual marginals are well-understood geometric variables, while the joint distribution is tractable.

Local: The aim of the local phase is to turn the discrete multivariate geometric distribution into a vector of continuous variables; the random variable S_i is drawn independently for each coordinate i and makes the coordinate continuous. Normalization by α_i allows for the marginal distribution of Z_i to be identical for all i.

The reason that the truncated *exponential* random variable S_i is combined with the multivariate *geometric* random variable X is because the exponential random variable is precisely the continuous analog of the geometric random variable. Hence, their sum enjoys nice properties. Moreover, the tail bounds of both geometric and exponential variables are fast-decaying, allowing for promising tail bounds for the overall objective.

The algorithm also reveals a few key differences between the bipartite perfect matching problem and the bipartite rounding problem. For example, the multivariate geometric distribution is symmetric with respect to the rate vector ρ in bipartite rounding, but this does not hold for the perfect bipartite matching problem. Moreover, we are unable to set the density of each random variable S_i independently in the perfect matching problem without accounting for the global matching constraint.

With this intuition for both the problem and the algorithm, we now present the mathematical formalism which this algorithm guarantees.

Proposition 3.1. *Each random variable Z_i satisfies $Z_i \sim \mathrm{Exp}(1)$.*

We omit the proof because it is straightforward and refer the reader to Proposition 2.1 in [3]. The idea is to compute the density of Z_i explicitly by separately marginalizing over X_i and S_i. The property that every Z_i has a well-understood marginal distribution that is the same across all coordinates i plays a central role in the bipartite rounding problem.

We now establish the negative-correlation property of the $\{Z_i\}_i$. This is a natural consequence of the construction of the multivariate geometric vector X.

Proposition 3.2. *For any indices $i_1, i_2 \in \{1, 2, \ldots, d\}$ with $i_1 \neq i_2$ and $\rho_{i_1}, \rho_{i_2} < 1$ and any values $q_1, q_2 \in (-\infty, 1)$, we have*

$$\mathbb{E}\left[e^{q_1 Z_{i_1} + q_2 Z_{i_2}}\right] = \frac{1}{(1-q_1)(1-q_2)} \cdot \frac{(1-\rho_{i_1})^{q_1} + (1-\rho_{i_2})^{q_2} + \rho_{i_1} + \rho_{i_2} - 2}{(1-\rho_{i_1})^{q_1}(1-\rho_{i_2})^{q_2} + \rho_{i_1} + \rho_{i_2} - 1}.$$

Before we give the proof, we remark on why this is a negative-correlation result. Because $Z_{i_1}, Z_{i_2} \sim \mathrm{Exp}(1)$, we have

$$\mathbb{E}\left[e^{q_1 Z_{i_1}}\right]\mathbb{E}\left[e^{q_2 Z_{i_2}}\right] = \frac{1}{(1-q_1)(1-q_2)}.$$

For well-chosen q_1 and q_2, the multiplicative parameter $\frac{(1-\rho_{i_1})^{q_1} + (1-\rho_{i_2})^{q_2} + \rho_{i_1} + \rho_{i_2} - 2}{(1-\rho_{i_1})^{q_1}(1-\rho_{i_2})^{q_2} + \rho_{i_1} + \rho_{i_2} - 1}$ is strictly less than one; this gives the strong negative correlation result.

Proof. Without loss of generality, let $i_1 = 1$ and $i_2 = 2$. The result is clear if either ρ_1 or ρ_2 is in $\{0, 1\}$, so we assume $\rho_1, \rho_2 \in (0, 1)$. Let $c_i = \alpha_i q_i$ and again let $L_i = X_i + S_i$. By definition,

$$\mathbb{E}\left[e^{q_1 Z_1 + q_2 Z_2}\right] = \mathbb{E}\left[e^{c_1 L_1 + c_2 L_2}\right] = \mathbb{E}\left[e^{c_1 S_1} \cdot e^{c_2 S_2} \cdot e^{c_1 X_1 + c_2 X_2}\right].$$

Since S_1 and S_2 are independent and both variables are independent from (X_1, X_2),

$$\mathbb{E}\left[e^{c_1 S_1} \cdot e^{c_2 S_2} \cdot e^{c_1 X_1 + c_2 X_2}\right] = \mathbb{E}\left[e^{c_1 S_1}\right]\mathbb{E}\left[e^{c_2 S_2}\right]\mathbb{E}\left[e^{c_1 X_1 + c_2 X_2}\right]. \tag{3.1}$$

We omit the rest of the proof as it is entirely computational and involves calculating each term explicitly (and calculating the joint distribution of $c_1 X_1 + c_2 X_2$). We refer the reader to Lemma 2.1 in [3].

We remark on where the negative correlation factor comes from. The equality

$$\mathbb{E}\left[e^{q_1 Z_1 + q_2 Z_2}\right] = \mathbb{E}\left[e^{c_1 S_1}\right]\mathbb{E}\left[e^{c_2 S_2}\right]\mathbb{E}\left[e^{c_1 X_1 + c_2 X_2}\right]$$

makes this evident. The variables S_1 and S_2 are independent from the setup with (X_1, X_2) and are thus cleanly separated on the right-hand side; because of their independence, the variables S_i on their own do not contribute a power savings in the correlation. The key fact is that X_1 and X_2 satisfy nonpositive correlation properties; this allows for the third term $\mathbb{E}[e^{c_1 X_1 + c_2 X_2}]$ to be less than $\mathbb{E}[e^{c_1 X_1}]\mathbb{E}[e^{c_2 X_2}]$ by a multiplicative factor (under certain properties on the q_i). Combining these with the other joint distributions gives us the strong negative correlation result. □

The last property we need is that the variables $(Z_1, Z_2, \ldots, Z_d)$ generated by CorrelatedExponential satisfy a stronger condition than nonpositive correlation called *negative association.*

Definition 3.3. A set of variables $(Y_1, \ldots, Y_k)$ is *negatively associated* if, for any disjoint $A_1, A_2 \subseteq \{1, \ldots, k\}$ of indices and any increasing functions f_1, f_2,

$$\mathbb{E}[f_1(\{Y_i \mid i \in A_1\}) f_2(\{Y_j \mid j \in A_2\})] \leq \mathbb{E}[f_1(\{Y_i \mid i \in A_1\})]\mathbb{E}[f_2(\{Y_j \mid j \in A_2\})].$$

We now have the following result.

Proposition 3.4. *The set $(Z_1, Z_2, \ldots, Z_d)$ generated by* CorrelatedExponential *is negatively associated.*

We omit the proof of this proposition as it relies heavily on machinery developed by [5, 8] and is not directly related to the theory of dependent rounding. A brief outline is that it is possible to show via standard techniques that the set $(X_1, X_2, \ldots, X_d)$ generated during the global phase of the algorithm is negatively associated; the result then follows because each Z_i is an increasing function of X_i and S_i and because the $\{S_i\}_i$ are independent from the $\{X_i\}_i$.

As an immediate correlation of the set $(Z_1, Z_2, \ldots, Z_d)$ being negatively associated, we conclude that

$$\mathbb{E}\left[\prod_{i=1}^{d} Z_i\right] \leq \prod_{i=1}^{d} \mathbb{E}[Z_i].$$

(This is again due to results from [5, 8].) Expressions involving such variables often are amenable to swapping *products* with *expectations* while preserving the direction of the original inequality.

Therefore, CorrelatedExponenitial generates a set of random variables that satisfies property (III) of *strong negative local correlation* and property (IV) of *general nonpositive correlation.* These properties are important for the overall bipartite rounding algorithm.

4. Applications to Bipartite Rounding

In this subsection, we analyze the CorrelatedExponential algorithm applied to bipartite rounding on graphs with negative correlation. The goal of the problem is to round a fractional vector $y \in \mathbb{R}^m$ to an integral one.

More precisely, assume that we have a fractional solution y which satisfies

$$\begin{aligned} 0 \leq y_i &\leq 1 && \forall\, 1 \leq i \leq m, \\ \sum_{\mathsf{e} \in \delta(v)} y_{\mathsf{e}} &\leq 1 && \forall\, v \in V. \end{aligned}$$

We generate a random vector $Y = (Y_1, Y_2, \ldots, Y_m)$ with $Y_i \in \{0, 1\}$ for all i and that satisfies properties (I), (II), (III), and (IV). These properties are important in establishing Chernoff-like tail bounds on the objective function; this allows one to obtain better *approximation ratios* on problems that can be solved using bipartite graphs, such as the machine-scheduling problem.

We briefly remark on the reason the ideas of CorrelatedExponential are a good fit for the bipartite rounding problem. Bipartite graphs contain both global and local structure: in addition to the overall *global* bipartition $U \cup V$, it is useful to think of the set of vertices $u \in U$ which are connected to a particular $v \in V$ and to analyze *local* induced subgraphs of the overall graph. For the randomized rounding problem, property (I) (i.e., *preservation of constraints* given by $\sum_{\mathsf{e} \in \delta(v)} Y_e \leq 1$ for all right nodes $v \in V$) lends itself to a natural interpretation of each vertex $v \in V$ *choosing* an edge $\mathsf{e} \in \delta(v)$ for which to set to one.

On the other hand, this formulation clearly deviates from the constraints of the perfect matching problem, which require that no two chosen edges share the same left node. This global matching constraint is not considered by the randomized rounding scheme, which is purely local, and hence is not guaranteed to hold.

Based on this formulation, however, we can adopt a similar idea to the setup of the *stable* bipartite matching problem. In particular, it is natural to think of each left node u proposing a possible value to each right node $v \in \Gamma(u)$; each right node v then chooses between the values received from the left vertices v is adjacent to. To obtain negative-correlation properties, it is natural to apply an algorithm with such properties to the edge neighborhood of every left node $u \in U$. When this subroutine is the algorithm CorrelatedExponential, we precisely recover the algorithm DepRound presented in [3].

Formally assume that our fractional solution $y \in \mathbb{R}^m$ satisfies the stricter properties

$$\begin{aligned} 0 < y_i &< 1 && \forall\, 1 \leq i \leq m, \\ \sum_{\mathsf{e} \in \delta(v)} y_{\mathsf{e}} &= 1 && \forall\, v \in V. \end{aligned} \tag{4.1}$$

This assumption can be made by removing edges with weights in $\{0, 1\}$ and adding "dummy" edges for each $v \in V$ which has $\sum_{\mathsf{e} \in \delta(v)} y_{\mathsf{e}} < 1$ in the original y. With these additional constraints, we present DepRound. Like CorrelatedExponential, DepRound takes in a rate vector $\rho \in \mathbb{R}^m$ with the same properties in addition to the fractional solution y as inputs. A distinction here is that the DepRound algorithm sets $u^* = \arg\min_{u \in \Gamma(v)}$ independently for every right node $v \in V$. The extra constraints in the perfect matching problem, however, suggests that we cannot perform a similar assignment in the perfect matching problem.

We first state the theorem regarding the behavior of DepRound, then prove the properties and define the necessary terms in sequential order.

Algorithm 2 DepRound

procedure DepRound(ρ, y) ▷ where $\rho \in [0,1]^m$ and $\sum_{i=1}^m \rho_i \leq 1$.
  for each left node $u \in U$ **do**
    Construct the vector $\rho^{(u)} \coloneqq \rho|_{\delta(u)}$, where $\rho|_{\delta(u)}$ is a size-$|\delta(u)|$ vector consisting only of the coordinates of ρ corresponding to edges in $\delta(u)$.
    Construct $Z^{(u)} \to$ CorrelatedExponential $\left(\rho^{(u)}\right)$.
  Construct $Z \in \mathbb{R}^m$, indexed by (u,v), by $Z_{(u,v)} = Z^{(u)}_v$.
  for each right node $v \in V$ **do**
    Choose the neighbor $u^* \coloneqq \arg\min_{u \in \Gamma(v)} Z_u / y_u$.
    Set $Y_{u^*} = 1$ and $Y_{u'} = 0$ for all $u' \in \delta(v)$ such that $u' \neq u^*$.
  return Y.

Theorem 4.1. *Algorithm DepRound satisfies the following properties.*

(I) (Preservation of constraints) *For each right node $v \in V$, we have* $\sum_{\mathsf{e} \in \delta(v)} Y_{\mathsf{e}} \leq 1$ *with probability* 1.

(II) (Preservation of expectation) *For each edge* $\mathsf{e} \in E$*, we have* $\mathbb{E}[Y_{\mathsf{e}}] = y_{\mathsf{e}}$.

(III) (General nonpositive correlation) *For any* stable *edge set* $\mathcal{E} \subseteq E$*, we have*

$$\mathbb{E}\left[\prod_{\mathsf{e} \in L} Y_{\mathsf{e}}\right] \leq \prod_{\mathsf{e} \in L} y_{\mathsf{e}}.$$

(IV) (Strong local negative correlation) *For any pair of edges* $\mathsf{e}_1 = (u, v_1)$ *and* $\mathsf{e}_2 = (u, v_2)$ *that share a common left node u, we have*

$$\mathbb{E}[Y_{\mathsf{e}_1} Y_{\mathsf{e}_2}] \leq \phi_{\mathsf{e}_1,\mathsf{e}_2} \cdot y_{\mathsf{e}_1} y_{\mathsf{e}_2},$$

where

$$\phi_{\mathsf{e}_1,\mathsf{e}_2} = \frac{(1-\rho_{\mathsf{e}_1})^{1-1/y_{\mathsf{e}_1}} + (1-\rho_{\mathsf{e}_2})^{1-1/y_{\mathsf{e}_2}} + \rho_{\mathsf{e}_1} + \rho_{\mathsf{e}_2} - 2}{(1-\rho_{\mathsf{e}_1})^{1-1/y_{\mathsf{e}_1}}(1-\rho_{\mathsf{e}_2})^{1-1/y_{\mathsf{e}_2}} + \rho_{\mathsf{e}_1} + \rho_{\mathsf{e}_2} - 1}.$$

The key equality (3.1) is necessary the strong local negative correlation property to hold. In the perfect matching problem, (3.1) does not necessarily hold for every pair of random variables because of the global matching constraint that no two chosen edges share the same left node. This is a fundamental difference between the two problems.

We now prove the properties in this theorem.

Proposition 4.2 (Preservation of Constraints)**.** *For each right node $v \in V$, we have* $\sum_{\mathsf{e} \in \delta(v)} Y_{\mathsf{e}} \leq 1$ *with probability* 1.

Proof. For each right node $v \in V$, the values v receives are from distinct left nodes u since G is simple. Therefore, the variables $\{Z_{\mathsf{e}} \mid \mathsf{e} \in \delta(v)\}$ are mutually independent. Moreover, CorrelatedExponential ensures that each variable in $\{Z_{\mathsf{e}} \mid \mathsf{e} \in \delta(v)\}$ has distribution Exp(1). Since they are independent and $\delta(v)$ is finite, the probability of any two values Z_{e}/y_e and $Z_{\mathsf{e}'}/y_{\mathsf{e}'}$ being equal is zero. This directly implies that the arg min u^* term in DepRound is unique with probability one. The result follows. ☐

This statement is one of the main reasons that we require our rounding scheme to be *continuous* and that working directly with the discrete multivariate geometric distribution is more involved.

Proposition 4.3 (Preservation of Expectation)**.** *For each* $\mathsf{e} \in E$*, we have* $\mathbb{E}[Y_{\mathsf{e}}] = y_{\mathsf{e}}$.

Proof. Since Y_{e} is a $\{0,1\}$-indicator variable, we show that $\Pr[Y_{\mathsf{e}} = 1] = y_{\mathsf{e}}$. To see this, let v be the right node corresponding to e, and consider the variables $\{Z_{\mathsf{e}'} \mid \mathsf{e}' \in \delta(v)\}$. We established in the proof of Proposition 4.2 that $\{Z_{\mathsf{e}'} \mid \mathsf{e}' \in \delta(v)\}$ is a set of independent random variables, each of which is exponential with rate one. Therefore, the set of random variables $\mathcal{S} = \left\{ \frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}} \mid \mathsf{e}' \in \delta(v) \right\}$ is a mutually independent set of exponential random variables which have rates $\{y_{\mathsf{e}'} \mid \mathsf{e}' \in \delta(v)\}$. Hence, the probability that $Y_{\mathsf{e}} = 1$ is the probability that e is the winner of the exponential race between the exponential variables with the aforementioned rates. By the standard properties of exponential random variables and (4.1), we have $\Pr[Y_{\mathsf{e}} = 1] = \dfrac{y_{\mathsf{e}}}{\sum_{\mathsf{e}' \in \delta(v)} y_{\mathsf{e}'}} = y_{\mathsf{e}}$, as required. □

The first two properties were fairly immediate consequences of our dependent rounding scheme. This illustrates the utility of having tractable marginal distributions for each individual variable in the dependent context. This setup also gives insight into the duality of the structure of bipartite graphs: despite the heavily dependent rounding we implement in CorrelatedExponential, a key property we used is that the values a particular right node v sees behave independently. We can then use these "pockets of independence" to establish results about local edge variables. This notably does not hold in the perfect bipartite matching problem since we must preserve the global structure of each *left* node also only having one incident right node.

For the remaining two properties, we use the global structure CorrelatedExponential provides. In CorrelatedExponential, we have nonpositive correlation across *all* variables due to negative association; when we call independent instances of CorrelatedExponential per left node, we would like to know the corresponding graphical structure in G this property extends to. These structures are called *stable edge sets*.

A stable edge set is defined in terms of an equivalence relation $\sim$ on E. Given two edges $\mathsf{e} = (u, v)$ and $\mathsf{e}' = (u', v')$ in E, we write $\mathsf{e} \sim \mathsf{e}'$ if and only if e is disjoint from e' (i.e., both $u \neq u'$ and $v \neq v'$) and at least of (u', v) and (u, v') is an edge in E. (In other words, two disjoint edges $\mathsf{e} = (u, v)$ and $\mathsf{e}' = (u', v')$ satisfy $\mathsf{e} \not\sim \mathsf{e}'$ if and only if the subgraph induced by (u, v, u', v') only has the two edges (u, v) and (u', v').)

Definition 4.4. A set $\mathcal{E} \subseteq E$ of edges in G is *stable* if and only if $\mathsf{e} \not\sim \mathsf{e}'$ for *every* pair $(\mathsf{e}, \mathsf{e}')$ of distinct edges in $\mathcal{E}$.

The key property of stable edge sets we use is the following.

Lemma 4.5. *Let $\mathcal{E} \subseteq E$ be a stable edge set, and let $\mathcal{E}' \subseteq E \setminus \mathcal{E}$ be the set of edges which share a right endpoint with an edge in $\mathcal{E}$. Let $\mathcal{E}_L$ and $\mathcal{E}'_L$ denote the set of left nodes which are endpoints of the edges of $\mathcal{E}$ and $\mathcal{E}'$, respectively. Then, $\mathcal{E}_L \cap \mathcal{E}'_L = \varnothing$.*

Proof. Assume for the sake of contradiction that there is a left node $u \in U$ which is in both $\mathcal{E}_L$ and $\mathcal{E}'_L$. By definition, there exist distinct vertices $v, v' \in V$ such that $(u, v) \in \mathcal{E}$ and $(u, v') \in \mathcal{E}'$. Since $(u, v') \in \mathcal{E}'$, we know that there is a $u' \in U$ with $u' \neq u$ such that $(u', v') \in \mathcal{E}$ by definition. However, we can now conclude that $(u, v) \sim (u', v')$ since $(u, v') \in \mathcal{E}' \subseteq E$. This contradicts the stability of $\mathcal{E}$. □

Intuitively, to get effective information on the joint distribution $\{Y_{\mathsf{e}}\}_{\mathsf{e}}$, it is helpful to think about the *conditional expectation* $\mathbb{E}\left[\prod_{\mathsf{e} \in \mathcal{E}} Y_{\mathsf{e}} \mid Z_{\mathcal{E}}\right]$ for sets $\mathcal{E} \subseteq E$. Conditioned on $Z_{\mathcal{E}}$, the only variables that both influence $\prod_{\mathsf{e} \in \mathcal{E}} Y_{\mathsf{e}}$ and whose distributions are affected by the conditioning are the variables corresponding to edges e' that share a right node with some edge in $\mathcal{E}$. Lemma 4.5 implies that when $\mathcal{E}$ is stable, these edges all come from different left nodes than that in $\mathcal{E}$; since the DepRound algorithm is parameterized by independent CorrelatedExponential calls to each left node, these distributions are also unaffected. (In the perfect bipartite matching problem, this is not necessarily true.) This leads us to get information on such conditional expectations when $\mathcal{E}$ is stable; more formally, we have the following result.

Proposition 4.6. *Let $\mathcal{E} \subseteq E$ be a stable set. Then*

$$\mathbb{E}\left[\prod_{\mathsf{e} \in \mathcal{E}} Y_{\mathsf{e}} \,\middle|\, Z_{\mathcal{E}}\right] \leq \prod_{\mathsf{e} \in \mathcal{E}} e^{(1 - 1/y_{\mathsf{e}}) \cdot Z_{\mathsf{e}}} \tag{4.2}$$

Proof. Note that if two edges in $\mathcal{E}$ share a right node, then $\prod_{\mathsf{e} \in \mathcal{E}} Y_{\mathsf{e}} = 0$ with probability one. Henceforth, assume that no two edges in $\mathcal{E}$ share a right node.

As we previously remarked, consider what happens when we condition on the entire set $Z_{\mathcal{E}}$. If we again define $\mathcal{E}'$ to be the set of edges which share a right endpoint with an edge in $\mathcal{E}$, we know from the above lemma that the stability of $\mathcal{E}$ guarantees that the distributions of the variables corresponding to edges in $\mathcal{E}'$ are not affected by the conditioning on $Z_{\mathcal{E}}$.

Since $\prod_{\mathsf{e}\in\mathcal{E}} Y_{\mathsf{e}}$ takes values in $\{0,1\}$, the result is equivalent to showing that we have $\Pr\left[\prod_{\mathsf{e}\in\mathcal{E}} Y_{\mathsf{e}} = 1\right] \leq \prod_{\mathsf{e}\in\mathcal{E}} e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}$. This product is one if and only if the values of Y_{e} for all $\mathsf{e} \in \mathcal{E}$ equal one. In particular, we can write (by definition)

$$\Pr\left[\bigwedge_{\mathsf{e}\in\mathcal{E}} Y_{\mathsf{e}} = 1\right] = \Pr\left[\bigwedge_{(u,v)\in\mathcal{E}} \frac{Z_{(u,v)}}{y_{(u,v)}} = \min_{\mathsf{e}'\in\delta(v)} \frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}}\right].$$

For the next result, we will need the result that $\{Z_{\mathsf{e}}\}_{\mathsf{e}\in E}$ are negatively associated; this follows as a direct corollary of the negative association in CorrelatedExponential. Because the event $\frac{Z_{(u,v)}}{y_{(u,v)}} = \min_{\mathsf{e}'\in\delta(v)} \frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}}$ is an *increasing function* of the random variables $Z_{\delta(v)\setminus\mathcal{E}}$ (i.e. the values of $Z_{\delta(v)\setminus\mathcal{E}}$ increasing also increases the probability of the event) and because the sets $\delta(v) \setminus \mathcal{E}$ are pairwise disjoint (as all right nodes in $\mathcal{E}$ are distinct in the case we're considering), the results regarding negatively-associated variables in [5, 8] directly show that

$$\Pr\left[\bigwedge_{(u,v)\in\mathcal{E}} \frac{Z_{(u,v)}}{y_{(u,v)}} = \min_{\mathsf{e}'\in\delta(v)} \frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}}\right] \leq \prod_{(u,v)\in\mathcal{E}} \Pr\left[\frac{Z_{(u,v)}}{y_{(u,v)}} = \min_{\mathsf{e}'\in\delta(v)} \frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}}\right].$$

Now, consider a fixed edge $\mathsf{e} \in \mathcal{E}$. When conditioned on $Z_{\mathcal{E}}$, the value of $\frac{Z_{\mathsf{e}}}{y_{\mathsf{e}}}$ is a fixed constant, however, the values of $\frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}}$ remain unaffected by our work above. In particular, we know that these are precisely mutually independent exponential random variables with rates $y_{\mathsf{e}'}$, hence, we also know that $\min_{\mathsf{e}'\in\delta(v)} \frac{Z_{\mathsf{e}'}}{y_{\mathsf{e}'}}$ is an exponential random variable with rate $\sum_{\mathsf{e}'\in\delta(v)\setminus\{e\}} y_{\mathsf{e}'} = 1 - y_{\mathsf{e}}$. It is then straightforward to compute that the probability that this variable is larger than the fixed value $\frac{Z_{\mathsf{e}}}{y_{\mathsf{e}}}$ is equal to $e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}$; this establishes the claim. □

Proposition 4.6 is necessary to establish both correlation properties of DepRound. Essentially, the result states that when *conditioned* on the values taken by a stable edge set, there is a small probability that all edges in that set are chosen.

Proposition 4.7 (Global Nonpositive Correlation). *For any stable edge set $\mathcal{E} \subseteq E$,*

$$\mathbb{E}\left[\prod_{\mathsf{e}\in\mathcal{E}} Y_{\mathsf{e}}\right] \leq \prod_{\mathsf{e}\in\mathcal{E}} y_{\mathsf{e}}.$$

Proof. Integrating (4.2) over all possible values of $Z_{\mathcal{E}}$ yields $\mathbb{E}\left[\prod_{\mathsf{e}\in\mathcal{E}} Y_{\mathsf{e}}\right] \leq \mathbb{E}_{Z_{\mathcal{E}}}\left[\prod_{\mathsf{e}\in\mathcal{E}} e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}\right]$.

Recall that the set $\{Z_{\mathsf{e}}\}_{\mathsf{e}\in E}$ are negatively associated, so we may swap *products* with *expectations* while preserving the inequality. Here, we give a similar simplification; formally, the negative association of $\{Z_{\mathsf{e}}\}_{\mathsf{e}}$ and the fact that $e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}$ is a decreasing function of Z_{e} implies that

$$\mathbb{E}_{Z_{\mathcal{E}}}\left[\prod_{\mathsf{e}\in\mathcal{E}} e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}\right] \leq \prod_{\mathsf{e}\in\mathcal{E}} \mathbb{E}_{Z_{\mathsf{e}}}\left[e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}\right]$$

Since each variable Z_{e} is an exponential random variable with parameter 1, it is straightforward to compute the inner expectation as exactly equal to y_{e} by integrating over the density. Therefore,

$$\mathbb{E}\left[\prod_{\mathsf{e}\in\mathcal{E}} Y_{\mathsf{e}}\right] \leq \mathbb{E}_{Z_{\mathcal{E}}}\left[\prod_{\mathsf{e}\in\mathcal{E}} e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}\right] \leq \prod_{\mathsf{e}\in\mathcal{E}} \mathbb{E}_{Z_{\mathsf{e}}}\left[e^{(1-1/y_{\mathsf{e}})\cdot Z_{\mathsf{e}}}\right] \leq \prod_{\mathsf{e}\in\mathcal{E}} y_{\mathsf{e}},$$

as required. □

We are now ready to show the central negative-correlation result that DepRound guarantees. As with the corresponding result in CorrelatedExponential, this follows as a corollary.

Proposition 4.8 (Strong Local Negative Correlation)**.** *For any pair of edges* $\mathsf{e}_1 = (u, v_1)$ *and* $\mathsf{e}_2 = (u, v_2)$ *which share a common left node* u*, we have*

$$\mathbb{E}[Y_{\mathsf{e}_1} Y_{\mathsf{e}_2}] \leq \phi_{\mathsf{e}_1,\mathsf{e}_2} \cdot y_{\mathsf{e}_1} y_{\mathsf{e}_2},$$

where

$$\phi_{\mathsf{e}_1,\mathsf{e}_2} = \frac{(1-\rho_{\mathsf{e}_1})^{1-1/y_{\mathsf{e}_1}} + (1-\rho_{\mathsf{e}_2})^{1-1/y_{\mathsf{e}_2}} + \rho_{\mathsf{e}_1} + \rho_{\mathsf{e}_2} - 2}{(1-\rho_{\mathsf{e}_1})^{1-1/y_{\mathsf{e}_1}}(1-\rho_{\mathsf{e}_2})^{1-1/y_{\mathsf{e}_2}} + \rho_{\mathsf{e}_1} + \rho_{\mathsf{e}_2} - 1}.$$

Proof. We fix a crucial typo in [3] which claims that the set $\mathsf{S} = \{\mathsf{e}_1, \mathsf{e}_2\}$ is a stable edge set. The set S is not stable because e_1 and e_2 are not disjoint. However, the same logic from our proof in Proposition 4.6 holds for S; the key point is that when we condition on Z_{e_1} and Z_{e_2}, the set of edges S' which have a right endpoint in $\{v_1, v_2\}$ and which are not in S do not have left endpoint u because the graph G is simple. Hence, the same proof from Proposition 4.6 holds for S and we can conclude that the result applies to S, as well.

Now, Proposition 4.6 directly implies that $\mathbb{E}[Y_{\mathsf{e}_1} Y_{\mathsf{e}_2} \mid Z_{\mathsf{e}_1}, Z_{\mathsf{e}_2}] \leq e^{(1-1/y_{\mathsf{e}_1})\cdot Z_{\mathsf{e}_1} + (1-1/y_{\mathsf{e}_2})\cdot Z_{\mathsf{e}_2}}$. Moreover, both Z_{e_1} and Z_{e_2} were generated by a single call to CorrelatedExponential because their corresponding edges share a left node; applying the strong correlation result we derived in the previous subsection to Z_{e_1} and Z_{e_2} with parameters $q_{\mathsf{e}_1} = 1 - \frac{1}{y_{\mathsf{e}_1}}$ and $q_{\mathsf{e}_2} = 1 - \frac{1}{y_{\mathsf{e}_2}}$ yields $\mathbb{E}\left[e^{(1-1/y_{\mathsf{e}_1})\cdot Z_{\mathsf{e}_1} + (1-1/y_{\mathsf{e}_2})\cdot Z_{\mathsf{e}_2}}\right] \leq \phi_{\mathsf{e}_1,\mathsf{e}_2} \cdot y_{\mathsf{e}_1} y_{\mathsf{e}_2}$ for the value of $\phi_{\mathsf{e}_1,\mathsf{e}_2}$ in the statement. Hence,

$$\mathbb{E}[Y_{\mathsf{e}_1} Y_{\mathsf{e}_2}] = \mathbb{E}[\mathbb{E}[Y_{\mathsf{e}_1} Y_{\mathsf{e}_2} \mid Z_{\mathsf{e}_1}, Z_{\mathsf{e}_2}]] \leq \mathbb{E}\left[e^{(1-1/y_{\mathsf{e}_1})\cdot Z_{\mathsf{e}_1} + (1-1/y_{\mathsf{e}_2})\cdot Z_{\mathsf{e}_2}}\right] \leq \phi_{\mathsf{e}_1,\mathsf{e}_2} \cdot y_{\mathsf{e}_1,\mathsf{e}_2}.$$

This completes the proof of the theorem. ∎

5. Conclusion

We surveyed the dependent bipartite graph rounding schemes of Rohewedder, Rouhani, and Wennmann [7] and Harris [3]. Specifically, we discussed Rohewedder, Rouhani, and Wennmann's dependent rounding scheme that improved the approximation to the bipartite perfect matching problem previously provided by Gandhi et al. [2] while also satisfying additional concentration constraints on predefined linear functions, which prove useful when applying bipartite perfect matching to solve problems such as the Budgeted Santa Claus Problem.

In the regime of general bipartite graphs (without perfect matching constraints), Harris [3] developed a dependent rounding algorithm that generates negatively-correlated random variables using the theory of negatively-correlated exponential random variables. The properties (I), (II), (III), and (IV) ensure tight tail concentration bounds of the objective function, which are amenable to problems that can be solved using bipartite graphs. In particular, this dependent rounding scheme improves the approximation ratio for the machine-scheduling problem. This scheme, however, cannot be directly applied to the bipartite perfect matching problem because the additional constraints in the bipartite perfect matching problem prohibits the algorithm from establishing negative correlation between the variables.

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