Recitation/Discussion 6

Exponential, Erlang, and Normal Distributions

accompaniment to MATH 170E: Introduction to Probability and Statistics 1: Probability
at the University of California, Los Angeles

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In this recitation, we will introduce and solve problems with exponential, Erlang, and normal distributions.

Exit Survey

Please fill out the exit survey.

This is so I can get feedback and know what to focus on next time!

Exponential Distributions

Recall that in a Poisson process the number of arrivals in a time interval follows a Poisson distribution. However, the waiting time between successive occurrences is also a random variable. It is continuous and follows the exponential distribution. We first consider the waiting time for the *first* arrival. We say $X \sim \text{Exp}(\lambda)$ if X follows the *exponential distribution with parameter* $\lambda > 0$.

Definition 1. Let $X \sim \text{Exp}(\lambda)$. Then X follows the pdf

$$f(t) = \lambda e^{-\lambda t}$$

For some motivation on this pdf, let us calculate the cdf (cumulative density function). Let X be the random variable for the waiting time for the first arrival, and let $F(t) = P(X \leq t)$ be the cdf. Then

$$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - P(N[0,t] = 0) = 1 - e^{-\lambda t}$$

By the fundamental theorem of calculus, we take the derivative to obtain

This is a fast way to calculate the pdf: Find the cdf first and take the derivative.

$$f(t) = F'(t) = \lambda e^{-\lambda t},$$

which is the required pdf.

Erlang Distributions

Rather than concentrating specifically on the first arrival, what if we want to consider the waiting time of the nth arrival, where $n \geq 1$? Such random variables follow the Erlang distribution. We say $X \sim \text{Erl}(n, \lambda)$ if X follows the *Erlang distribution with paramter n and* λ.

Definition 2. Let $X \sim \text{Erl}(n, \lambda)$. Then X follows the pdf

$$f(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}.$$

We follow the same calculation. Let X be the random variable for the waiting time for the nth arrival, and let $F(t) = P(X \leq t)$ be the cdf. Then

$$\begin{aligned} F(t) &= P(X \leq t) \\ &= 1 - P(X > t) \\ &= 1 - P(N[0,t] < n) \\ &= 1 - e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right) \end{aligned}$$

is the cdf of X. We take the derivative to find the pdf, which is

$$\begin{aligned} f(t) &= F'(t) \\ &= \lambda e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right) \\ &\quad - e^{-\lambda t}\left(\frac{\lambda}{1!} + \frac{2\lambda(\lambda t)}{2!} + \cdots + \frac{\lambda(n-1)(\lambda t)^{n-2}}{(n-1)!}\right) \\ &= \lambda e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right) \\ &\quad - \lambda e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \cdots + \frac{(\lambda t)^{n-2}}{(n-2)!}\right) \\ &= \lambda \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \\ &= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}. \end{aligned}$$

Note that the exponential distribution is a special case of the Erlang distribution when $n = 1$: $X \sim \text{Exp}(\lambda)$ if and only if $X \sim \text{Erl}(1, \lambda)$.

There is also a generalization of the Erlang distribution known as the *Gamma distribution*.

We have the following forms for the expected value and variance of exponential and Erlang distributions.

Proposition 3. *Let $X \sim \text{Erl}(n, \lambda)$. Then*

$$\mathbb{E}[X] = \frac{n}{\lambda} \quad \textit{and} \quad \text{Var}(X) = \frac{n}{\lambda^2}.$$

To prove these identities, we use the pdf that we just calculated, so

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} t f(t)\, dt = \int_{-\infty}^{\infty} \frac{\lambda^n t^n e^{-\lambda t}}{(n-1)!}\, dt = \frac{n}{\lambda},$$

Omitting a lot of details. Good for you to go through and verify.

$$\text{Var}(X) = \int_{-\infty}^{\infty} t^2 f(t)\, dt - \left(\int_{-\infty}^{\infty} t f(t)\, dt\right)^2 = \frac{n}{\lambda^2}.$$

Also omitting a lot of details. You should verify.

You should verify both integrals directly using integration by parts.

Exercise 1. Suppose a hacker hacked into your email and is sending emails according to a Poisson process with rate λ per hour.

(a) What is the expected time for the hacker to send the first email?

(b) What is the expected time for the hacker to send the tenth email?

(c) What is the probability that the hacker sends the fourth email between hour 5 and 7?

Solution. (a) Let X be the time for the hacker to send the first email. This is an exponential distribution (i.e., Erlang distribution with $n = 1$) with parameter λ, so its expected value is

$$\mathbb{E}[X] = \frac{n}{\lambda} = \frac{1}{\lambda}.$$

(b) Let X be the time for the hacker to send the tenth email. This is an Erlang distribution with $n = 10$, so its expected value is

$$\mathbb{E}[X] = \frac{n}{\lambda} = \frac{4}{\lambda}.$$

(c) Let X be the time for the hacker to send the fourth email. This is an Erlang distribution with $n = 4$, so the probability is

$$P(5 \leq X \leq 7) = \int_5^7 \frac{\lambda^4 t^3 e^{-\lambda t}}{3!}\, dt.$$

We can compute this integral, but we can also realize we already have the antiderivative: the cdf $F(t)$ when $n = 4$ is

$$F(t) = 1 - e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!}\right).$$

Therefore, we also have

$$\begin{aligned} P(5 \leq X \leq 7) &= F(7) - F(5) \\ &= e^{-5\lambda}\left(1 + 5\lambda + \frac{(5\lambda)^2}{2} + \frac{(5\lambda)^3}{3!}\right) \\ &\quad - e^{-7\lambda}\left(1 + 7\lambda + \frac{(7\lambda)^2}{2} + \frac{(7\lambda)^3}{3!}\right). \end{aligned}$$

Normal Distributions

The normal distribution is the typical “bell curve." It has two parameters: where its center is and how wide it is, governed by the mean and variance respectively. We say that $X \sim \mathcal{N}(\mu, \sigma)$ if X follows the normal distribution with mean μ and variance σ^2.

Definition 4. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then X follows the pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

μ is the *mean* and σ^2 is the variance; σ is known as the standard deviation.

The *standard normal distribution* is the normal distribution when $\mu = 0$ and $\sigma^2 = 1$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

In fact, we can obtain the original pdf to the pdf of the standard normalized distribution using the change of variables $x \mapsto (x-\mu)/\sigma$. This motivates the following derivation, which justifies that f is a pdf.

Theorem 5. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Then

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Proof. We change the variables of integration by letting $z = (x-\mu)/\sigma$, so

$$I := \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

We consider I^2. This is

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dx\,dy. \end{aligned}$$

We convert to polar coordinates. Let $x = r\cos\theta$, $y = r\sin\theta$, so the Jacobian is

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Then the determinant of J is

$$\det J = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Under this change of variables we have

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1,$$

so $I = 1$ since the integrand $f(x) \geq 0$ is always nonnegative. ∎

Proposition 6. *Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then*

$$\mathbb{E}[X] = \mu \quad \textit{and} \quad \operatorname{Var}(X) = \sigma^2.$$

You should actually sit down and prove it using moment generating functions (see below).

To prove this, we use the power of the moment generating function.

Definition 7. The *moment generating function* of a pdf $f(x)$ is given by

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

Proposition 8. *If X has pdf $f(x)$, then*

$$\mathbb{E}[X] = M'(0) \quad \textit{and} \quad \operatorname{Var}(X) = M''(0) - M'(0)^2.$$

You should actually sit down and prove it.

Exercise 2. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ satisfies $Z \sim \mathcal{N}(0, 1)$.

Solution. The cdf of Z is

$$\begin{aligned} P(Z \le z) &= P\left(\frac{X - \mu}{\sigma} \le z\right) \\ &= P(X \le \sigma z + \mu) \\ &= \int_{-\infty}^{\sigma z + \mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \end{aligned}$$

Now use the change of variables $w = (x - \mu)/\sigma$ to obtain

$$P(Z \le z) = \int_{-\infty}^{\sigma z + \mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

which is the cdf of the standard normal distribution.

A cool threshold: for a normal distribution,

- 99.7% of the mass lies within three standard deviations from the mean;
- 95% of the mass lies within two standard deviations from the mean; and
- 68% of the mass lies within one standard deviation from the mean.

In other words, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\begin{aligned} P(\mu - \sigma \le X \le \mu + \sigma) &\approx 0.6827, \\ P(\mu - 2\sigma \le X \le \mu + 2\sigma) &\approx 0.9545, \\ P(\mu - 3\sigma \le X \le \mu + 3\sigma) &\approx 0.9973. \end{aligned}$$