

Recitation/Discussion 5

Binomial, Geometric, Negative Binomial, Poisson, and Continuous Distributions

accompaniment to MATH 170E: Introduction to Probability and Statistics 1: Probability
at the University of California, Los Angeles

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In this recitation, we will review discrete random variables. Then, we will introduce continuous random variables.

Exit Survey

Please fill out the exit survey.

This is so I can get feedback and know what to focus on next time!

Discrete Distributions

The following table summarizes the data.

X	$P(X = k)$	$\mathbb{E}[X]$	$\text{Var}(X)$
$X \sim \text{Bin}(n, p)$	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
$X \sim \text{Geom}(p)$	$(1-p)^{k-1} p$	$1/p$	$(1-p)/p^2$
$X \sim \text{NB}(r, p)$	$\binom{k-1}{r-1} (1-p)^{k-r} p^r$	r/p	$r(1-p)/p^2$
$X \sim \text{Pois}(\lambda)$	$\lambda^k e^{-\lambda} / k!$	λ	λ

Binomial Distributions

Definition 1. The *binomial distribution* with parameters n and p is the discrete probability distribution of the number of successes of n independent trials, with each trial resulting in a success with probability p . If X is a random variable that follows the binomial distribution with n trials with probability p , we write $X \sim \text{Bin}(n, p)$. Each experiment is known as a *Bernoulli trial*.

Proposition 2. Let $X \sim \text{Bin}(n, p)$. Then X follows the pmf

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k \in \{0, \dots, n\}$.

Proof. Let $k \in \{0, \dots, n\}$. For a fixed choice of k successes out of n experiments, there are $n - k$ failures, with a resulting probability of $p^k (1-p)^{n-k}$. There are $\binom{n}{k}$ ways to choose the positions of k successes, so the total probability is $\binom{n}{k} p^k (1-p)^{n-k}$. ■

An easy way to remember this: there are k successes out of n experiments, of which there are $\binom{n}{k}$ such arrangements.

Proposition 3. If $X \sim \text{Bin}(n, p)$, then $\mathbb{E}[X] = np$.

Proof. Let $\mathbb{I}_i$ be the indicator variable for success on the i th experiment. Then $X = \mathbb{I}_1 + \cdots + \mathbb{I}_n$, so by linearity of expectation

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{I}_1 + \cdots + \mathbb{I}_n] = \mathbb{E}[\mathbb{I}_1] + \cdots + \mathbb{E}[\mathbb{I}_n] = \underbrace{p + \cdots + p}_{n \text{ times}} = np,$$

which finishes the proof. ■

Proposition 4. If $X \sim \text{Bin}(n, p)$, then $\text{Var}[X] = np(1 - p)$.

Proof 1. Let $\mathbb{I}_i$ be the indicator variable for success on the i th experiment. Then $X = \mathbb{I}_1 + \cdots + \mathbb{I}_n$. Since $\mathbb{I}_1, \dots, \mathbb{I}_n$ **are independent**, we can write

$$\text{Var}(X) = \text{Var}(\mathbb{I}_1 + \cdots + \mathbb{I}_n) = \text{Var}(\mathbb{I}_1) + \cdots + \text{Var}(\mathbb{I}_n) = np(1 - p)$$

since $\text{Var}(\mathbb{I}_i) = \mathbb{E}[\mathbb{I}_i^2] - \mathbb{E}[\mathbb{I}_i]^2 = \mathbb{E}[\mathbb{I}_i] - \mathbb{E}[\mathbb{I}_i]^2 = p - p^2 = p(1 - p)$. ■

Proof 2. We just expand $X^2 = (\mathbb{I}_1 + \cdots + \mathbb{I}_n)^2$ to get

$$\begin{aligned} \mathbb{E}[X^2] &= \mathbb{E}[(\mathbb{I}_1 + \cdots + \mathbb{I}_n)^2] = \mathbb{E}\left[\sum_{i=1}^n \mathbb{I}_i^2 + \sum_{i \neq j} \mathbb{I}_i \mathbb{I}_j\right] \\ &= \sum_{i=1}^n \mathbb{E}[\mathbb{I}_i] + \sum_{i \neq j} \mathbb{E}[\mathbb{I}_i \mathbb{I}_j] = np + n(n-1)p^2 \end{aligned}$$

$$\text{so } \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = np + n(n-1)p^2 - (np)^2 = np(1 - p). \quad \blacksquare$$

Proof 3. We recall that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, so we need only calculate $\mathbb{E}[X^2]$. This is just a computation:

$$\begin{aligned} \mathbb{E}[X^2] &= \sum_{k=0}^n k^2 \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n kn \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^n k \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\ &= np \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^j (1-p)^{n-1-j} \\ &= np \left(\sum_{j=0}^{n-1} j \binom{n-1}{j} p^j (1-p)^{n-1-j} + \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \right) \\ &= np \left(p \sum_{j=1}^{n-1} j \binom{n-1}{j} p^{j-1} (1-p)^{(n-1)-(j-1)} + \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \right) \\ &= np \left(p \sum_{j=1}^{n-1} (n-1) \binom{n-2}{j-1} p^{j-1} (1-p)^{(n-2)-(j-1)} + \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \right) \\ &= np \left((n-1)p + (1-p) \right) = np(1-p), \end{aligned}$$

$$\text{so } \text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = np(1-p) - (np)^2 = np(1-p). \quad \blacksquare$$

This is an excellent application of indicator random variables.

Again, expressing a sum of variables as a sum of indicator variables will be a recurring idea.

This is the slicker proof. It turns out that variance is also linear **under independence**, which by assumption we have.

I like this proof too.

Linearity of expectation, $\mathbb{I}_i^2 = \mathbb{I}_i$, $n(n-1)$ pairs $i \neq j$ (why?), and $\mathbb{E}[\mathbb{I}_i \mathbb{I}_j] = p^2$ (why?)

You should verify each step of the computation.

By definition

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

$k = 0$ term is 0 and reindexing

$j = k - 1$ and $m = n - 1$

Splitting sum into two

$j = 0$ term is 0 and reindexing

Binomial theorem

Geometric Distributions

Definition 5. The *geometric distribution* with parameter p is the discrete probability distribution of the number of trials needed for one success, with each trial resulting in a success with probability p . If X is a random variable that follows the geometric distribution with probability p , we write $X \sim \text{Geom}(p)$.

Proposition 6. Let $X \sim \text{Geom}(p)$. Then X follows the pmf

$$P(X = k) = (1 - p)^{k-1}p$$

for $k \in \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$. If the first success is on the k th experiment, then the first $k - 1$ are failures. The latter happens with probability $(1 - p)^{k-1}$, and the k th experiment occurs with success with probability p . The total probability is thus $(1 - p)^{k-1}p$. ■

Proposition 7. If $X \sim \text{Geom}(p)$, then $\mathbb{E}[X] = 1/p$.

Proof 1. The first trial either succeeds with probability p or fails with probability $1 - p$. If it fails, the remaining expected number of trials until a success is identical to the original expected number (if we just disregard the first trial) since all the trials are independent. Therefore,

$$\mathbb{E}[X] = p + (1 - p)(1 + \mathbb{E}[X]),$$

which gives $\mathbb{E}[X] = 1/p$. ■

Proof 2. Recall the sum of the geometric series is

$$a + \sum_{k=1}^{\infty} ar^k = \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

for $|r| < 1$. Differentiating with respect to r gives

$$\sum_{k=1}^{\infty} kar^{k-1} = \frac{a}{(1 - r)^2}$$

for $|r| < 1$. We just calculate the expected value

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} k(1 - p)^{k-1}p \\ &= \frac{p}{(1 - (1 - p))^2} \\ &= \frac{1}{p}, \end{aligned}$$

as required. ■

An easy way to remember this: $k - 1$ failures before success.

This should make sense from an intuitive standpoint: You should expect two coin flips for a head, or six dice rolls for a 5.

This is the slicker proof.

This is justified by the uniform convergence of power series.

Here $a = 1$ and $r = 1 - p$.

Proposition 8. If $X \sim \text{Geom}(p)$, then $\text{Var}(X) = (1 - p)/p^2$.

Proof. Recall that

$$\sum_{k=1}^{\infty} kar^{k-1} = \frac{a}{(1-r)^2}$$

for $|r| < 1$. Therefore,

$$\sum_{k=1}^{\infty} kar^k = \frac{ar}{(1-r)^2}$$

Differentiating with respect to r gives

$$\sum_{k=1}^{\infty} k^2 ar^{k-1} = \frac{a(1-r)^2 + 2ar(1-r)}{(1-r)^4} = \frac{a(1-r) + 2ar}{(1-r)^3} = \frac{a(1+r)}{(1-r)^3}$$

for $|r| < 1$. We just calculate

$$\begin{aligned} \mathbb{E}[X^2] &= \sum_{k=1}^{\infty} k^2 P(X = k) \\ &= \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p \\ &= \frac{p(1 + (1-p))}{(1 - (1-p))^3} \\ &= \frac{2-p}{p^2}. \end{aligned}$$

Therefore,

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2},$$

as required. ■

Negative Binomial Distributions

Definition 9. The *negative binomial distribution* with parameters r and p is the discrete probability distribution of the number of trials needed for r successes, with each trial resulting in a success with probability p . If X is a random variable that follows the negative binomial distribution for r successes with probability p , we write $X \sim \text{NB}(r, p)$.

Also known as a *Pascal distribution*.

Note that $\text{Geom}(p) = \text{NB}(1, p)$.

Proposition 10. Let $X \sim \text{NB}(r, p)$. Then X follows the pmf

$$P(X = k) = \binom{k-1}{r-1} (1-p)^{k-r} p^r$$

for $k \geq r$.

An easy way to remember this: The k th trial is a success, and the remaining $r - 1$ successes are out of the last $k - 1$ trials.

Proof. Let $k \geq r$. If the r th success is on the k th experiment, then the previous $r - 1$ successes occurred in the previous $k - 1$ experiments. There are $\binom{k-1}{r-1}$ such arrangements, each with probability $p^{r-1}(1-p)^{(k-1)-(r-1)} = p^{r-1}(1-p)^{k-r}$. Accounting for the r th success which happens with probability p , the total probability is thus $\binom{k-1}{r-1}p^r(1-p)^{k-r}$. ■

Proposition 11. If $X \sim \text{NB}(r, p)$, then $\mathbb{E}[X] = r/p$.

Proof. For each $i \in \{1, \dots, r\}$ let Y_i be the random variable denoting the number of trials needed to obtain the i th success after the $(i - 1)$ th success. Then

$$X = Y_1 + \dots + Y_r$$

counts the number of trials needed for r successes, so

$$\mathbb{E}[X] = \mathbb{E}[Y_1 + \dots + Y_r] = \mathbb{E}[Y_1] + \dots + \mathbb{E}[Y_r] = \underbrace{1/p + \dots + 1/p}_{r \text{ times}} = r/p,$$

since each Y_i is a geometric random variable. ■

Proposition 12. If $X \sim \text{NB}(r, p)$, then $\text{Var}(X) = r(1-p)/p^2$.

Proof. Since $Y_1, \dots, Y_r$ are independent, we can write

$$\text{Var}(X) = \text{Var}(Y_1 + \dots + Y_r) = \text{Var}(Y_1) + \dots + \text{Var}(Y_r) = \frac{r(1-p)}{p^2},$$

since each Y_i is a geometric random variable. ■

Note that this is r times as much as the expectation of a geometric random variable. This might suggest a linearity of expectation proof where we have r random variables, one for each success.

This is a common trick.

This is r times the variance of a geometric random variable.

Poisson Distributions

Definition 13. The *Poisson distribution* with parameter λ is the discrete probability distribution of the number of events occurring in a fixed time interval assuming that

- the number of events occur at a fixed rate λ and
- the number of occurrences in nonoverlapping intervals are independent.

If X is a random variable that follows the Poisson distribution with parameter λ , we write $X \sim \text{Pois}(\lambda)$, and its pmf is of the form for $k \geq 0$

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Proposition 14. If $X \sim \text{Pois}(\lambda)$, then $\mathbb{E}[X] = \lambda$.

Proof. By definition,

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^{\lambda} = \lambda \end{aligned}$$

by the Taylor series for e^x . ■

Note here we are defining the pmf.

Recall that the Taylor series for e^x is $e^x = \sum_{k=0}^{\infty} x^k/k!$.

Proposition 15. If $X \sim \text{Pois}(\lambda)$, then $\text{Var}(X) = \lambda$.

Proof. We calculate $\mathbb{E}[X(X-1)]$. By linearity of expectation, we have

$$\mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mathbb{E}[X] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \text{Var}(X).$$

The calculation is

$$\begin{aligned} \mathbb{E}[X(X-1)] &= \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!} \\ &= \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = \lambda^2 e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2. \end{aligned}$$

$$\text{Then } \text{Var}(X) = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda. \quad \blacksquare$$

It turns out that a Poisson random variable can be approximated by a binomial random variable. If λ is the number of successes in a fixed time period of length 1, then we may divide this into n time periods of length $1/n$. Each subinterval has a probability of $\lambda \cdot (1/n) = \lambda/n$ of success. Therefore, $\text{Pois}(\lambda) \approx \text{Bin}(n, \lambda/n)$, and the approximation gets better as $n \rightarrow \infty$ (or equivalently as $h \rightarrow 0$). To make this precise, note that for any $k \geq 0$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P(\text{Bin}(n, \lambda/n) = k) &= \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n(n-1) \cdots (n-(k-1))}{n^k} \right) \\ &\quad \times \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left(\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-(k-1)}{n} \right) \\ &\quad \times \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left(1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \right) \\ &\quad \times \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} \cdot 1 \\ &= \frac{\lambda^k e^{-\lambda}}{k!} \\ &= P(\text{Pois}(\lambda) = k). \end{aligned}$$

Exercises

Exercise 1. A basketball player makes each free throw with $9/10$ probability. Suppose the player attempts 20 free throws.

- (i) What is the probability that the player makes all 20 free throws?
- (ii) What is the probability that the player makes 15 free throws?
- (iii) What is the probability that the player makes at least 10 free throws?
- (iv) What is the number of free throws the player is expected to make?
- (v) What is the variance in the number of free throws the player makes?

Solution. Let's create a random variable

$$X = \text{\#successful free throws.}$$

- (i) This is $P(X = 20) = (9/10)^{20}$; each free throw has a $9/10$ chance of success, so 20 successes happen with probability $(9/10)^{20}$.
- (ii) This is $P(X = 15) = \binom{20}{15}(9/10)^{15}(1/10)^5$. For 15 free throws, there must be exactly 15 successes and 5 failures, and there are $\binom{20}{15}$ ways to arrange 15 successes in 20 attempts.
- (iii) This is $P(10 \leq X \leq 20) = \sum_{i=10}^{20} \binom{20}{i}(9/10)^i(1/10)^{20-i}$.
- (iv) When X is binomial, $\mathbb{E}[X] = np$, where n is the number of trials and p is the probability of success. Here $n = 20$ and $p = 9/10$, so $\mathbb{E}[X] = 20 \cdot 9/10 = 18$.
- (v) When X is binomial, $\text{Var}(X) = np(1 - p)$, where n is the number of trials and p is the probability of success. Here $n = 20$ and $p = 9/10$, so $\text{Var}(X) = 20 \cdot 9/10 \cdot 1/10 = 9/5$.

You should know how to prove this. Set up indicator variables for each success, and apply linearity of expectation.

Exercise 2. A factory produces items independently, each of which has probability $1/4$ of being defective. A batch contains 20 items.

- (i) Find the probability that at least 3 items are defective and at most 6 are defective.
- (ii) Given that at least one item is defective, find the probability exactly two are defective. There should be no summation $\sum$ in your answer.
- (iii) Suppose the batch is accepted if at most 2 items are defective. Find the probability that the batch is accepted.
- (iv) The factory inspects items one at a time until it finds the first defective item. Let N denote the number of items inspected, including the first defective item. Find $P(N = 5)$. Explain why N does not follow a binomial distribution. What distribution does it follow?
- (v) Now suppose the defect probability p is unknown. You observe that exactly 4 of the 20 items are defective. Find the value of p that maximizes the likelihood of this observation.

Solution. Let's create a random variable

$$X = \text{\#defective items.}$$

(i) This is $P(3 \leq X \leq 6) = \sum_{i=3}^6 \binom{20}{i} (1/4)^i (3/4)^{20-i}$.

(ii) We calculate

$$P(X = 2 \mid X \geq 1) = \frac{P(X = 2 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X = 2)}{P(X \geq 1)}.$$

But $P(X = 2) = \binom{20}{2} (1/4)^2 (3/4)^{18}$ and

$$P(X \geq 1) = 1 - P(X = 0) = 1 - (3/4)^{20},$$

so the final answer is

$$P(X = 2 \mid X \geq 1) = \frac{\binom{20}{2} (1/4)^2 (3/4)^{18}}{1 - (3/4)^{20}}.$$

(iii) This is $P(X \leq 2) = \sum_{i=0}^2 \binom{20}{i} (1/4)^i (3/4)^{20-i}$.

(iv) For $N = 5$, the first four items must be acceptable and the fifth is defective. Therefore, $P(N = 5) = (3/4)^4 (1/4)$. A binomial random variable counts the number of successes in a fixed number of independent trials. Here N follows a geometric distribution where the process stops as soon as a defective item is found.

(v) Here $P(X = 4) = \binom{20}{4} p^4 (1-p)^{16}$. To find a p that maximizes this, we can take the derivative to get

$$\binom{20}{4} (4p^3(1-p)^{16} - 16p^4(1-p)^{15}) = \binom{20}{4} p^3(1-p)^{15}(4-20p),$$

which is zero when $p = 0$, $p = 1$, or $p = 1/5$. One can check that the maximum probability is attained when $p = 1/5$.

The derivative is positive for $p < 1/5$ and negative for $p > 1/5$, or you can use your favorite second derivative test.

Exercise 3. You flip a weighted coin that lands on heads with probability $1/5$ and on tails with probability $4/5$.

- (i) What is the probability that you see your first heads at the 6th flip?
- (ii) Find the probability that you flip the coin at least 5 times before getting heads.
- (iii) Find the expected number of attempts you need to flip your coin until you get heads.
- (iv) What is the variance in the number of attempts you need to flip until you get heads?
- (v) Given that the first two flips were tails, what is the probability that the first heads is on the 5th flip?

Solution. Let's create a random variable

$X = \text{\#of coin flips until first heads.}$

(i) Here $P(X = 6) = (4/5)^5(1/5)$.

(ii) Here

$$P(X \geq 5) = \sum_{i=5}^{\infty} \left(\frac{4}{5}\right)^i \left(\frac{1}{5}\right) = \frac{1}{5} \cdot \frac{(4/5)^5}{1 - 4/5} = \frac{(4/5)^5}{5 - 4} = \left(\frac{4}{5}\right)^5.$$

(iii) When X is geometric, $\mathbb{E}[X] = 1/p$, where p is the probability of success. Here $p = 1/5$, so $\mathbb{E}[X] = 1/(1/5) = 5$.

(iv) When X is geometric, $\text{Var}(X) = (1 - p)/p^2$, where p is the probability of success. Here $p = 1/5$, so $\text{Var}(X) = (1 - 1/5)/(1/5)^2 = 20$.

(v) This is

$$\begin{aligned} P(X = 5 \mid \text{first two flips are tails}) &= \frac{P(X = 5 \text{ and first two flips are tails})}{P(\text{first two flips are tails})} \\ &= \frac{(4/5)^4(1/5)}{(4/5)^2} = \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right). \end{aligned}$$

Exercise 4. You flip a weighted coin that lands on heads with probability $1/3$ and on tails with probability $2/3$.

- (i) Find the probability that the third head appears on the 7th flip.
- (ii) Find the probability that at least 7 flips are needed to obtain the third head.
- (iii) Find the expected number of attempts you need to flip your coin until you get three heads.
- (iv) What is the variance in the number of attempts you need to flip until you get three heads.
- (v) Given that the first two flips were tails, what is the probability that the third heads is on the 7th flip?

Solution. Let's create a random variable

$X = \text{\#of coin flips until third heads.}$

- (i) For the third head to appear on the 7th flip, the first six flips must contain exactly two heads, and the seventh flip must be a head. Thus

$$P(X = 7) = \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right).$$

- (ii) Here the event $N \geq 7$ means that the third head has not appeared in the first six flips. Equivalently, those 6 flips contain at most 2 heads.

$$P(X = 7) = P(\text{at most 2 heads in 6 flips}) = \sum_{i=0}^2 \binom{6}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{6-i}.$$

- (iii) When X is negative binomial, $\mathbb{E}[X] = r/p$, where p is the probability of success and r is the number of required successes. Here $p = 1/3$ and $r = 3$, so $\mathbb{E}[X] = 3/(1/3) = 9$.

- (iv) When X is negative binomial, $\text{Var}(X) = r(1-p)/p^2$, where p is the probability of success and r is the number of required successes. Here $p = 1/3$ and $r = 3$, so $\text{Var}(X) = 3(1 - 1/3)/(1/3)^2 = 18$.

- (v) This is

$$P(X = 7 \mid \text{first two flips are tails}) = \frac{P(X = 7 \text{ and first two flips are tails})}{P(\text{first two flips are tails})}.$$

This quantity is

$$\begin{aligned} & \frac{P(\text{tail on flips 1 and 2})P(2 \text{ heads for flips 3, 4, 5, 6})P(\text{heads on flip 7})}{(2/3)^2} \\ &= \frac{(2/3)^2 \binom{4}{2} (1/3)^2 (2/3)^2 (1/3)}{(2/3)^2} = \frac{8}{81}. \end{aligned}$$

Exercise 5. A university help desk receives an average of 4 support requests per hour. Assume requests arrive according to a Poisson process, with a constant arrival rate and independent numbers of arrivals in disjoint time intervals.

- (i) Find the probability that the help desk receives exactly 6 requests in one hour.
- (ii) Find the probability that it receives at most 2 requests in one hour.
- (iii) Given that 3 requests arrived in the first hour, find the probability that exactly 2 requests arrive in the next hour.
- (iv) Find the expectation and variance of the number of requests received in a chosen hour.

Solution. Let's create a random variable

$$X = \text{\#requests received in one hour.}$$

Since the average number of requests per hour is 4, here we choose $\lambda = 4$.

- (i) Here $P(X = 6) = e^{-4} \cdot 4^6 / 6!$.
- (ii) Here $P(X \leq 2) = e^{-4} (4^0 / 0! + 4^1 / 1! + 4^2 / 2!) = 13e^{-4}$.
- (iii) Since requests in disjoint time intervals are independent, the number of requests in the next hour is independent of the number in the first hour. So this is just $P(X = 2) = e^{-4} \cdot 4^2 / 2! = 8e^{-4}$.
- (iv) When X is Poisson, $\mathbb{E}[X] = \text{Var}(X) = \lambda = 4$.

Continuous Distributions

Now we shift our focus to continuous distributions. As the name suggests, this distribution describes random variables that are *continuous*. For example, if we want to analyze a random number between a and b , the random variable is

$$X = \text{number obtained between } a \text{ and } b.$$

Of course, X can behave differently according to the distribution of X . Now there is no notion of a pmf, but there is a notion of a *probability density function* (or pdf). This is motivated by the physical argument that if we integrate density across a dimension we obtain the mass.

- If we want our X to be uniformly distributed or have a *uniform distribution*, we want our pdf to be

$$f(x) = \frac{1}{b-a}$$

for all $x \in [a, b]$.

- Of course, X can have different distributions.

Definition 16. The *probability density function* (pdf) of a continuous random variable X is a function $f(x)$ that satisfies

- $f(x) > 0$ for all $x \in S$,
- $\int_S f(x) dx = 1$, and
- $P(X \in (a, b)) = \int_a^b f(x) dx$ for all $(a, b) \subseteq S$.

Compare this to the definition of a pmf of a discrete random variable, and how we replace sums with integrals.

Definition 17. The *probability mass function* (pmf) of a discrete random variable X is a function $f(x)$ that satisfies

- $f(x) > 0$ for all $x \in S$,
- $\sum_{x \in S} f(x) = 1$, and
- $P(X \in A) = \sum_{x \in A} f(x)$ for all $A \subseteq S$.

Similarly, the definitions of expected value and variance change analogously for continuous random variables.

Definition 18. If X is a continuous random variable with pdf f over a sample space $[a, b]$, then its *expected value* is

$$\mathbb{E}[X] = \int_a^b xf(x) dx.$$

In fact, there is no way to program a computer to output a random number, because such an algorithm is by construction *deterministic*. Talk to me for more details.

Notice that these are exactly the axioms of probability.

Notice that these are exactly the axioms of probability.

Definition 19. If X is a continuous random variable with pdf f over a sample space $[a, b]$, then its *variance* is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_a^b (x - \mathbb{E}[X])^2 f(x) dx.$$

Compare this to the definitions of a discrete random variable.

Definition 20. If X is a continuous random variable with pmf f over a sample space S , then its *expected value* is

$$\mathbb{E}[X] = \sum_{x \in S} xf(x).$$

Definition 21. If X is a continuous random variable with pmf f over a sample space $[a, b]$, then its *variance* is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in S} (x - \mathbb{E}[X])^2 f(x).$$

Exercise 6. A bus arrives at a stop at a random time uniformly distributed between 0 and 20 minutes after you arrive. Let X be your waiting time in minutes.

- (a) Find the probability density function of X .
- (b) Find $\mathbb{E}[X]$.
- (c) Find $\text{Var}(X)$.
- (d) Find the probability that you wait more than 15 minutes.

Solution. (a) Here the pdf is $f(x) = 1/20$.

(b) We have

$$\mathbb{E}[X] = \int_0^{20} xf(x) dx = \int_0^{20} \frac{x}{20} dx = \left[\frac{x^2}{40} \right]_0^{20} = \frac{20^2}{40} - \frac{0^2}{40} = 10.$$

(c) We have

$$\begin{aligned} \text{Var}(X) &= \int_0^{20} (x - 10)^2 f(x) dx = \int_0^{20} \frac{(x - 10)^2}{20} dx \\ &= \left[\frac{(x - 10)^3}{60} \right]_0^{20} = \frac{10^3}{60} - \frac{(-10)^3}{60} = \frac{100}{3}. \end{aligned}$$

(d) Here

$$P(X > 15) = \int_{15}^{20} f(x) dx = \int_{15}^{20} \frac{1}{20} dx = \left[\frac{x}{20} \right]_{15}^{20} = \frac{1}{4}.$$

Exercise 7. A particle is confined to a one-dimensional infinite potential well on the interval $[0, L]$. Suppose its normalized wave function in the ground state is

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

for $x \in [0, L]$. According to Born's rule, the probability density of finding the particle at position x is $|\psi(x)|^2$.

- Find the probability density function $f(x)$ for the particle's position at x .
- Find the cumulative distribution function.
- Find the probability that the particle is found in the left half of the well.
- Find the probability that the particle is found in the middle third of the well.
- Find the probability that the particle is found within a distance $L/4$ of the center of the well.
- Find the expected position of the particle and its variance.

Solution. Let X be the position of the particle.

- According to the problem, the pdf is

$$f(x) = |\psi(x)|^2 = \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right).$$

- We calculate the antiderivative of $f(x)$ from 0 to x . This is

$$\begin{aligned} F(x) &= \int_0^x f(t) dt = \int_0^x \frac{2}{L} \sin^2\left(\frac{\pi t}{L}\right) dt = \frac{2}{\pi} \int_0^{\pi x/L} \sin^2(u) du & u = \pi t/L, \text{ so } du = \pi/L dt. \\ &= \frac{2}{\pi} \int_0^{\pi x/L} \frac{1 - \cos(2u)}{2} du = \frac{1}{\pi} \left[u - \frac{1}{2} \sin(2u) \right]_0^{\pi x/L} \\ &= \frac{1}{\pi} \left(\frac{\pi x}{L} - \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) \right). \end{aligned}$$

- This is

$$P(0 \leq X \leq L/2) = \int_0^{L/2} f(x) dx = F(L/2) - F(0) = 1/2.$$

- This is

$$P(L/3 \leq X \leq 2L/3) = F(2L/3) - F(L/3) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}.$$

(e) This is

$$P(L/4 \leq X \leq 3L/4) = F(3L/4) - F(L/4) = \frac{1}{2} + \frac{1}{\pi}.$$

(f) By definition we have

$$\begin{aligned} \mathbb{E}[X] &= \int_0^L x f(x) dx \\ &= \frac{2}{L} \int_0^L x \sin^2\left(\frac{\pi x}{L}\right) dx \\ &= \frac{2L}{\pi^2} \int_0^\pi u \sin^2 u du && u = \pi x/L, \text{ so } du = \pi/L dx \\ &= \frac{2L}{\pi^2} \int_0^\pi \frac{u(1 - \cos(2u))}{2} du \\ &= \frac{L}{\pi^2} \int_0^\pi u du - \frac{L}{\pi^2} \int_0^\pi u \cos(2u) du \\ &= \frac{L}{\pi^2} \frac{u^2}{2} \Big|_0^\pi - \frac{L}{\pi^2} \left(\left[\frac{u}{2} \sin(2u) \right]_0^\pi + \left[\frac{\cos(2u)}{4} \right]_0^\pi \right) \\ &= \frac{L}{2}. \end{aligned}$$

Now $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - L^2/4$. We have

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^L x^2 f(x) dx \\ &= \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx \\ &= \frac{2L^2}{\pi^3} \int_0^\pi u^2 \sin^2 u du && u = \pi x/L, \text{ so } du = \pi/L dx \\ &= \frac{2L^2}{\pi^3} \int_0^\pi \frac{u^2(1 - \cos(2u))}{2} du \\ &= \frac{L^2}{\pi^3} \int_0^\pi u^2 du - \frac{L^2}{\pi^3} \int_0^\pi u^2 \cos(2u) du \\ &= \frac{L^2}{\pi^3} \frac{u^3}{3} \Big|_0^\pi - \frac{L^2}{\pi^3} \left(\left[\frac{u^2 \sin(2u)}{2} \right]_0^\pi + \left[\frac{u \cos(2u)}{2} \right]_0^\pi - \left[\frac{\sin(2u)}{4} \right]_0^\pi \right) \\ &= \frac{L^2}{3} - \frac{L^2}{2\pi^2} = L^2 \left(\frac{1}{3} - \frac{1}{2\pi^2} \right), \end{aligned}$$

so

$$\text{Var}(X) = \mathbb{E}[X^2] - \frac{L^2}{4} = L^2 \left(\frac{1}{3} - \frac{1}{2\pi^2} \right) - \frac{L^2}{4} = L^2 \left(\frac{1}{12} - \frac{1}{2\pi^2} \right),$$

as required.