Recitation/Discussion 3

Discrete Random Variables and Expected Value

accompaniment to MATH 170E: Introduction to Probability and Statistics 1: Probability
at the University of California, Los Angeles

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In this recitation, we will introduce discrete random variables. Then, we will study properties of expected value.

Exit Survey

Please fill out the exit survey.

This is so I can get feedback and know what to focus on next time!

Discrete Random Variables

A random variable assigns a real number x to each element s of the sample space S. Despite its name, a random variable is *neither* random *nor* it an unknown value; rather, assigning a real number x to each element $s \in S$ exactly describes a function.

Unfortunately, this misnomer stuck because it associates a number to each outcome, each of which could be considered random.

Definition 1. A function $X\colon S \to \mathbb{R}$ is a *random variable* that assigns a real number $X(s) = x$ to each element $s \in S$.

Of course, this definition is very vague, as we can assign *any* real number to each element $s \in S$. This is precisely our prerogative, as

1. often the probabilities assigned to events are unknown, and
2. we want to make our random variable definition meaningful.

In this particular recitation we take X to be *discrete*, which means that S is finite or countable.

A set S is countable if and only if there is a bijection $S \to \mathbb{N}$. Intuitively, discrete sets S are sets that have isolated points, such as $\{1, 2, 3\}$ or $\mathbb{N}$ or $\mathbb{Z}$.

Example 1. Suppose that Sam flips two coins in a row. The sample space S is thus $S = \{\text{TT}, \text{TH}, \text{HT}, \text{HH}\}$. If Sam wants to analyze the number of heads, then a reasonable random variable definition is

$$X(\text{TT}) = 0, \qquad X(\text{TH}) = X(\text{HT}) = 1, \qquad X(\text{TT}) = 2.$$

Example 2. A census wants to analyze the number of people born each year. The sample space S is $S = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}$, and a reasonable random variable definition is

This is an example of a countable set.

$$X(n) = n$$

for $n \in S$.

An important type of a discrete random variable is the *indicator random variable* that is 1 whenever a desirable event happens and 0 otherwise.

The value 1 *indicates* that the event happened.

Example 3. Back to Sam and their coins, suppose Sam wants to analyze the event that they get only one head. Sam can use an indicator random variable with definition

$$X(\mathrm{TT}) = 0, \qquad X(\mathrm{TH}) = X(\mathrm{HT}) = 1, \qquad X(\mathrm{TT}) = 0.$$

The probability $P(X = x)$ is frequently denoted as $f(x)$ and is refered to as the *probability mass function* (pmf).

Definition 2. The *probability mass function* (pmf) of a discrete random variable X is a function $f(x)$ that satisfies

Notice that these are exactly the axioms of probability.

- $f(x) > 0$ for all $x \in S$,
- $\sum_{x \in S} f(x) = 1$, and
- $P(X \in A) = \sum_{x \in A} f(x)$ for all $A \subseteq S$.

If f is constant on S, we say that f is *uniform*.

Exercise 1. Suppose that we are rolling an 8-sided die numbered $\{1, \ldots, 8\}$ once and want to analyze the result. Define the sample space S, an appropriate definition of a random variable X with its possible values, and the probability mass function $f(x)$ of X. Is f uniform?

Exercise 2. Suppose there is an urn of 25 balls numbered $\{1, \ldots, 25\}$, and we reached in and grabbed three without replacement. We want to analyze the largest of the three balls. Define the sample space S, an appropriate definition of a random variable X with its possible values, and the probability mass function $f(x)$ of X. Is f uniform?

Expected Value

Now that we have a notion of a random variable, a natural question is asking what value the random variable takes on average. This motivates the definition of *expected value*.

Definition 3. Let X be a discrete random variable with pmf $f(x)$, then the *expected value of* $u(X)$ is

$$\mathbb{E}[u(X)] = \sum_{x \in S} u(x) f(x).$$

In essence, this is just a weighted average, where the weights are determined by the probabilities.

For example, let's determine the expected roll of an 8-sided die.

Exercise 3. Calculate Sam's expected value for their roll of an 8-sided die.

Exercise 4. In a lottery, there are 100 balls numbered $\{1, \ldots, 100\}$ are placed in a machine and six of them are drawn at random with replacement. Each lottery ticket cost 10 dollars and contains six numbers $\{1, \ldots, 100\}$, possibly the same.

- If all six numbers of your ticket match, then you win 1000 dollars!
- If only five numbers of your ticket match, then you win 100 dollars.

Define an appropriate random variable and find the expected value of your winnings.

The most important property regarding expectation is that it behaves linearly. This is called linearity of expectation.

This property is the quintessential tool in probability. This is so useful.

Theorem 4 (Linearity of expectation)**.** Let X be a discrete random variable with pmf $f(x)$, and let c and d be constants. Then

(i) $E(c) = c$;

(ii) $\mathbb{E}[cu(X)] = c\mathbb{E}[u(X)]$; and

(iii) $\mathbb{E}[cu_1(X) + du_2(X)] = c\mathbb{E}[u_1(X)] + d\mathbb{E}[u_2(X)]$.

In fact, this extends to any finite sum using induction:

$$\mathbb{E}\left[\sum_{i=1}^{n} c_i u_i(X)\right] = \sum_{i=1}^{n} c_i \mathbb{E}[u_i(X)].$$

Proof. (i) We have

$$\mathbb{E}[c] = \sum_{x \in S} cf(x) = c \sum_{x \in S} f(x) = c$$

since

$$\sum_{x \in S} f(x) = 1.$$

(ii) We have

$$\mathbb{E}[cu(X)] = \sum_{x \in S} cu(x)f(x) = c \sum_{x \in S} u(x)f(x) = c\mathbb{E}[u(X)].$$

(iii) We have

$$\begin{aligned} \mathbb{E}[cu_1(X) + du_2(X)] &= \sum_{x \in S} (cu_1(x)f(x) + du_2(x)f(x)) \\ &= \sum_{x \in S} cu_1(x)f(x) + \sum_{x \in S} du_2(x)f(x) \\ &= c\mathbb{E}[u_1(x)] + d\mathbb{E}[u_2(x)]. \end{aligned}$$

We have our result. ■

Exercise 5. It's your lucky day! You are going to flip 8 fair coins. If you flip n heads, then you win n dollars. What is your expected payout? Define an appropriate random variable.

What changes when the coins are not fair? Try this same exercise if you flip heads with probability $p \in [0, 1]$.

Hint. Use indicator variables $X_1, \ldots, X_8$ where X_i is the indicator variable for the ith flip.