

Directions. Complete all the problems. Your solutions to the problems should be submitted to Gradescope before the indicated due date above. Please follow rules regarding Gradescope submission as described in the syllabus. If you use any outside source in your solutions, you must cite the source.

Problem 1. 20 people are in an elevator, starting at floor 1. The elevator stops at each floor as it ascends. Suppose that each time the elevator arrives at a new floor, each person who is still on board will exit the elevator with probability $p \in (0, 1)$, independent of the other passengers. Note that no one exits at floor 1 and that no one ever enters the elevator.

- (a) Let A be the event that exactly 3 people get off at floor 2 and exactly 5 people get off at floor 3. Compute $P(A)$.

Final answer. $\frac{20!}{3! 5! 12!} p^8 (1 - p)^{29}$.

- (b) What is the expected number of people still on the elevator immediately after it departs the 4th floor?

Solution. A given person is still aboard after floor 4 exactly when they stayed at floors 2, 3, 4, which has probability $(1 - p)^3$. Thus, the number remaining is $\text{Binom}(20, (1 - p)^3)$, so its expected value is $20(1 - p)^3$.

Problem 2. You are a realtor selling a large house which has 5 keys, one for each of the doors into the house. All the keys look the same, so to open the front door you try the keys at random.

- (a) Find the pmf of the number of trials you will need to open the door, under the following alternative assumptions: (1) after an unsuccessful trial, you mark the corresponding key, so that you never try it again; and (2) at each trial you are equally likely to choose any key.

Solution. Let X be the number of trials.

(1) Without repetition, the correct key is equally likely to be in any of the 5 positions of the random order in which keys are tried, so $P(X = k) = \frac{1}{5}$ for $k = 1, \dots, 5$ (and $= 0$ for all other k).

(2) With replacement, each trial succeeds independently with probability $\frac{1}{5}$, so $X \sim \text{Geom}(1/5)$: $P(X = k) = \left(\frac{4}{5}\right)^{k-1} \frac{1}{5}$ for $k \geq 1$ (and 0 otherwise).

- (b) Repeat part (a) for the case where you were given one extra duplicate key for each of the 5 doors.

Final answer. (1) $P(X = k) = \frac{10 - k}{45}$, $k = 1, \dots, 9$ (and $= 0$ for all other k).

(2) $P(X = k) = \left(\frac{4}{5}\right)^{k-1} \frac{1}{5}$, $k = 1, 2, 3, \dots$ (and $= 0$ for all other k)

Problem 3. Consider the Bernoulli trial which consists of tossing two coins simultaneously. We repeat until a trial in which one coin lands heads while the other lands tails. One coin has probability $p \in (0, 1)$ of landing heads, while the other has probability $q \in (0, 1)$ of landing heads. All tosses are assumed to be independent.

Find the pmf, the expected value, and the variance of the number of trials.

Final answer. Let X be the number of trials. $X \sim \text{Geo}(p + q - 2pq)$ (and we can get all the requested information from this fact)

Problem 4. We model the number of times a neuron fires with a Poisson process. Suppose that on average the neuron fires $\lambda > 0$ times per second. We start at time $t = 0$ seconds.

- (a) Given that the neuron fires exactly 3 times in the first 2 seconds, find the conditional probability that the first arrival happened after 0.5 seconds.

Final answer. $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$.

- (b) What is the probability that the second firing happens after 4 seconds?

Final answer. $e^{-4\lambda}(1 + 4\lambda)$.

- (c) Given that the first firing happened within 1 second, what is the conditional probability that the second firing also happened within 1 second?

Final answer. $\frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}} = 1 - \frac{\lambda}{e^{\lambda} - 1}$.

Problem 5. A machine prints pages continuously and takes 1 minute to print 1 page. (So, page 1 is printed during the time interval $[0, 1)$, page 2 is printed during the time interval $[1, 2)$, etc.)

We model the occurrence of printing errors using a Poisson process with mean $\frac{1}{4}$ errors per minute. We call a page *unusable* if it contains 2 or more printing errors.

- (a) Compute the probability that the 2nd error made by the machine occurs on page 2.

Final answer. $\frac{5}{4}e^{-1/4} - \frac{3}{2}e^{-1/2}$.

- (b) Given that page 1 is unusable, what is the conditional probability that the 3rd error occurs after page 2?

Solution. Page 1 is unusable exactly when $N[0, 1) \geq 2$. The 3rd error occurs after page 2 exactly when $N[0, 2) \leq 2$.

$$\begin{aligned} P(N[0, 2) \leq 2 \mid N[0, 1) \geq 2) &= \frac{P(N[0, 2) \leq 2 \ \& \ N[0, 1) \geq 2)}{P(N[0, 1) \geq 2)} \\ &= \frac{P(N[0, 1) = 2 \ \& \ N[1, 2) = 0)}{P(N[0, 1) \geq 2)} = \frac{P(N[0, 1) = 2)P(N[1, 2) = 0)}{P(N[0, 1) \geq 2)} \\ &= \frac{\frac{e^{-1/4}}{32} \cdot e^{-1/4}}{1 - \frac{5}{4}e^{-1/4}} = \frac{e^{-1/2}}{32 \left(1 - \frac{5}{4}e^{-1/4}\right)} \end{aligned}$$

Note that we used the fact that $N[0, 1) = 2$ and $N[1, 2) = 0$ are independent events since they are about non-overlapping time intervals.

- (c) For this part, suppose the machine shuts down immediately after it finishes printing the 5th unusable page. Let X be the number of pages printed. Compute the expected value and variance of X .

Final answer. All the requested information can be obtained easily once one establishes that X has a negative binomial distribution:

$$X \sim \text{NB}(5, 1 - \frac{5}{4}e^{-1/4})$$

Recommended practice. The following exercises from the textbook are not assigned and do not have to be turned in. They are recommended to get more practice.

Section 2.4: 3, 12, 14

Section 2.6: 1, 4, 8

Section 2.7: 3, 8, 11

Most of the textbook exercises are very good, so you can always get more practice by working on exercises not mentioned in the above list.