

Directions. Complete all the problems. Your solutions to the problems should be submitted to Gradescope before the indicated due date above. Please follow rules regarding Gradescope submission as described in the syllabus. If you use any outside source in your solutions, you must cite the source.

Problem 1. Roll two fair dice, one red and one blue. Let X be the result of the red die, and let Y be the result of the blue die. Set $Z = X - Y$.

(a) Find the pmf of Z .

$$\text{Final answer. } P(Z = z) = \begin{cases} 1/36 & \text{if } z = 5 \text{ or } z = -5 \\ 2/36 & \text{if } z = 4 \text{ or } z = -4 \\ \vdots & \\ 5/36 & \text{if } z = 1 \text{ or } z = -1 \\ 6/36 & \text{if } z = 0. \end{cases}$$

(b) Find the pmf of Z^2 .

$$\text{Final answer. } P(Z^2 = z) = \begin{cases} 2/36 & \text{if } z = 5^2 \\ 4/36 & \text{if } z = 4^2 \\ \vdots & \\ 10/36 & \text{if } z = 1^2 \\ 6/36 & \text{if } z = 0. \end{cases}$$

(c) Find the expected value and variance of Z .

$$\text{Final answer. } E[Z] = 0, \text{Var}(Z) = \frac{2 \cdot 5^2 + 4 \cdot 4^2 + 6 \cdot 3^2 + 8 \cdot 2^2 + 10 \cdot 1^2}{36}$$

Problem 2. Let A and B be events such that $P(A \cup B) = 0.8$, $P(A) = 0.5$, and $P(B) = 0.4$. Find the expected values of the following random variables:

$$I_A \cdot I_B, \quad I_A + I_B, \quad 2I_A - I_B^2.$$

$$\text{Final answer. } E[I_A \cdot I_B] = .1, \quad E[I_A + I_B] = .9, \quad E[2I_A - I_B^2] = .6$$

Problem 3. You have a coin which lands heads with probability $2/3$. You toss that coin $n > 2$ times. Call tosses $i, i + 1$ a *matched pair* if they both landed on the same side.

- (a) Given that $1, 2$ is a matched pair, what is the conditional probability that $2, 3$ is a matched pair?

Final answer. $\frac{(2/3)^3 + (1/3)^3}{(2/3)^2 + (1/3)^2}$

- (b) Find the expected number of matched pairs. *Hint.* Use indicator random variables.

Solution. For $i = 1, \dots, n - 1$, let I_i be the indicator RV for the event that $i, i + 1$ is a matched pair. Let X be the number of matched pairs. Then,

$$X = I_1 + \dots + I_{n-1}.$$

For any $i = 1, \dots, n - 1$,

$$\mathbb{E}[I_i] = P(i, i + 1 \text{ is a matched pair}) = (2/3)^2 + (1/3)^2 = 5/9.$$

Thus,

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \dots + \mathbb{E}[I_{n-1}] = \frac{5(n-1)}{9}.$$

Problem 4.

- (a) Suppose X is a random variable that takes values $1, \dots, n$. Show that $\mathbb{E}[X] = \sum_{i=1}^n P(X \geq i)$. *Hint.* Start with an easy case like $n = 3$ or 4 and see if it gives you an idea for the general case.

Solution. This is shown by showing that the sum $\sum_{i=1}^n P(X \geq i)$ can

be rearranged into $\mathbb{E}[X]$ as follows:

$$\begin{aligned}
\sum_{i=1}^n P(X \geq i) &= P(X \geq 1) \\
&\quad + P(X \geq 2) \\
&\quad + P(X \geq 3) \\
&\quad \vdots \\
&\quad + P(X \geq n) \\
&= P(X = 1) + P(X = 2) + P(X = 3) + \cdots + P(X = n) \\
&\quad + P(X = 2) + P(X = 3) + \cdots + P(X = n) \\
&\quad + P(X = 3) + \cdots + P(X = n) \\
&\quad \vdots \\
&\quad + P(X = n) \\
&= P(X = 1) + 2P(X = 2) + 3P(X = 3) + \cdots + nP(X = n) \\
&= \mathbb{E}[X]
\end{aligned}$$

- (b) Does the more general statement hold that if X only takes positive integer values and $\mathbb{E}[X]$ exists, then $\mathbb{E}[X] = \sum_{i=1}^{\infty} P(X \geq i)$? You don't need to write a formal proof of your answer, but provide good reasoning.

Final answer. The more general statement is true. (The full proof relies on the fact that, for an absolutely convergent series, we can rearrange the summands in any way we like and we will still get the same sum.)

Problem 5. For a random variable with mean μ , $\mathbb{E}[(X - \mu)^3]$ is a measure of the skewness of the distribution of X .

Suppose X is a discrete random variable (so it is integer-valued) such that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^3]$ is finite. Suppose also that X is symmetric about its mean in the sense that $P(X = n) = P(X = -n)$ for $n = 1, 2, \dots$. Show that $\mathbb{E}[X^3] = 0$.

Solution. We have

$$\begin{aligned}
E[X^3] &= \sum_{n=-\infty}^{\infty} n^3 P(X = n) = \left(\sum_{n=1}^{\infty} (-n)^3 P(X = -n) \right) + 0P(X = 0) + \left(\sum_{n=1}^{\infty} n^3 P(X = n) \right) \\
&= \sum_{n=1}^{\infty} (-n^3) P(X = n) + \sum_{n=1}^{\infty} n^3 P(X = n) = - \sum_{n=1}^{\infty} n^3 P(X = n) + \sum_{n=1}^{\infty} n^3 P(X = n) = 0.
\end{aligned}$$

Recommended practice. The following exercises from the textbook are not assigned and do not have to be turned in. They are recommended to get more practice.

Section 2.1: 1, 2, 7, 8, 11

Section 2.2: 4, 5, 10

Section 2.3: 4, 6, 7, 9

Most of the textbook exercises are very good, so you can always get more practice by working on exercises not mentioned in the above list.