

Selected solutions and final answers. Below are full solutions and some final answers for selected problems. Things marked “final answer” give just that—the final answer. But, keep in mind that you are expected to show your work and justify your answers. Also keep in mind: it is always better for your learning to go through the entire problem-solving process than to just read someone else’s solution.

Directions. Complete all the problems. Your solutions to the problems should be submitted to Gradescope before the indicated due date above. Please follow rules regarding Gradescope submission as described in the syllabus. If you use any outside source in your solutions, you must cite the source.

Problem 1. For this problem, it will be very helpful to recall that for a real number x with $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Consider an experiment where we randomly select a positive integer. Our sample space is $S = \{1, 2, 3, \dots\}$. Suppose that our probability function P satisfies

$$P(\{n\}) = \frac{1}{2^n}, \quad \text{for } n = 2, 3, \dots$$

- (a) Use the given information about P and the properties of probability functions to find $P(1)$.

Final answer. $1/2$

- (b) Let $k \in S$. Compute the probability that we select a number $\geq k$.

Final answer. $1/2^{k-1}$

- (c) Let $k \in S$. Compute the probability that we select a number $\leq k$.

Final answer. $1 - \frac{1}{2^k}$

- (d) Compute the probability that we select an even number.

Final answer. $1/3$

Problem 2. There are n chairs placed around a round table for n guests to sit in.

- (a) Assume the chairs are distinguishable from each other, e.g., labeled 1 through n . How many distinct ways are there to seat the n guests?

Final answer. $n!$

- (b) Now assume the chairs are not distinguishable. You consider two seating arrangements to be the same if they only differ by a rotation of the table. With these assumptions, how many distinct seating arrangements are there?

Hint. The count from part (a) is an over-count for this problem. How many times does it over-count each arrangement?

Final answer. $(n - 1)!$

- (c) Now assume that you have two tables, each with n chairs to seat a total of $2n$ guests. The tables are distinguishable, e.g., one is red and one is blue. The chairs are not distinguishable, so two arrangements at a table that only differ by a rotation are considered to be the same. Under these assumptions, how many distinct seating arrangements are there?

Solution. First we have to pick which n people will sit at the red table. There are $\binom{2n}{n}$ ways to do this. (Then, we know the other n will sit at the blue table.)

Next, by part (b), there are $(n - 1)!$ ways to arrange the people at the red table.

Finally, by part (b) again there are $(n - 1)!$ ways to arrange the people at the blue table.

Multiplying these together, the final answer is $\binom{2n}{n}((n - 1)!)^2$.

- (d) Consider the same set up as in part (c), but now with the two tables indistinguishable. How many seating arrangements are there with this new assumption?

Hint. Again, part (c) is an over-count for this problem. How can you correct it?

Final answer. $\frac{1}{2} \binom{2n}{n} ((n - 1)!)^2$

Problem 3. You are to create a sign. You have wooden letters AABBBCCCC and you will use all of them in your sign. After you arrange the letters, you will also paint each letter one of 3 colors.

In how many ways can you create the sign?

Problem 4.

- (a) You flip a fair coin 6 times. What is the probability that you flip more heads than tails?

Solution. (a) All outcomes are equally likely because the coin is fair. We count the number of outcomes where there are more heads than tails:

number with 4 H's: $\binom{6}{4} = 15$

number with 5 H's: $\binom{6}{5} = 6$

number with 6 H's: 1

So the number of outcomes with more H's than T's is $15 + 6 + 1 = 22$. To find the probability of more H's than T's, we divide this number by the total number of outcomes:

$$\frac{22}{2^6}$$

- (b) Let n be an odd positive integer. Flip a fair coin n times. What is the probability that you flip more heads than tails?

Final answer. $1/2$. (You can establish this answer without having to actually count anything!)

Recommended practice. The following exercises from the textbook are not assigned and do not have to be turned in. They are recommended to get more practice.

Section 1.1: 1, 3, 4, 5, 11, 15

Section 1.2: 3, 5, 7, 9, 11, 17

Most of the textbook exercises are very good, so you can always get more practice by working on exercises not mentioned in the above list.