Math 170E **Midterm 1 Solutions**
UCLA, Fall 2026

1. The ten letters of the word

S T A T I S T I C S

(three S's, three T's, two I's, one A, one C) are arranged into a word (a string of letters).

(a) How many distinct words can be formed using all ten letters? For this part, you can just write down the answer. You do not need any justification.

Answer. $\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}$

(b) One of the distinct words from part (a) is chosen uniformly at random. Compute the probability that the two I's are *not* next to each other.

Answer. Let E be the event that the two I's are not next to each other. Then, E' is the event that the I's are next to each other.

Since the word is chosen uniformly at random, we can computing the probability using counting. To count the words in E', glue the two I's together into a single block "II". We are then arranging 9 objects: three S's, three T's, one block II, one A, and one C. The number of such arrangements is

$$|E'| = \binom{9}{3}\binom{6}{3}\binom{3}{1}\binom{2}{1}.$$

Therefore

$$P(E') = \frac{\binom{9}{3}\binom{6}{3}\binom{3}{1}\binom{2}{1}}{\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}}$$

and so

$$P(E) = 1 - P(E') = 1 - \frac{\binom{9}{3}\binom{6}{3}\binom{3}{1}\binom{2}{1}}{\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}}$$

(This happens to simplify to 4/5.)

2. Let X be a discrete random variable with $\mathbb{E}[X] = 2$ and $\mathrm{Var}(X) = 3$. Compute $\mathbb{E}[X(X-1)]$.

Answer. Since $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, we have

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = 3 + 2^2 = 7.$$

By linearity of expectation,

$$\mathbb{E}[X(X-1)] = \mathbb{E}[X^2] - \mathbb{E}[X] = 7 - 2 = 5.$$

3. Each of 8 students independently chooses one of 4 office hour time slots, with each slot equally likely.

(a) Let A_1 be the event that the first time slot is chosen by no one, and let A_2 be the event that the second time slot is chosen by no one. Are A_1 and A_2 independent? (Justify your answer with computations, not intuition.)

Answer. The event A_1 occurs exactly when each of the 8 students chooses one of the other 3 slots. Each student does this with probability 3/4, independently, so

$$P(A_1) = \left(\frac{3}{4}\right)^8, \qquad \text{and similarly} \qquad P(A_2) = \left(\frac{3}{4}\right)^8.$$

The event $A_1 \cap A_2$ occurs exactly when each student chooses one of the last 2 slots, so

$$P(A_1 \cap A_2) = \left(\frac{2}{4}\right)^8 = \left(\frac{1}{2}\right)^8.$$

On the other hand,

$$P(A_1)P(A_2) = \left(\frac{3}{4}\right)^{16} = \left(\frac{9}{16}\right)^8.$$

Since $\frac{1}{2} \neq \frac{9}{16}$, we have $P(A_1 \cap A_2) \neq P(A_1)P(A_2)$, so A_1 and A_2 are *not* independent.

(b) Compute the expected number of time slots that no student chooses.

Answer. For $i = 1, 2, 3, 4$, let A_i be the event that slot i is chosen by no one, and let I_i be the indicator random variable of A_i. The number of time slots that no student chooses is $N = I_1 + I_2 + I_3 + I_4$. As in part (a), $\mathbb{E}[I_i] = P(A_i) = (3/4)^8$ for each i. By linearity of expectation (which does not require independence),

$$\mathbb{E}[N] = \sum_{i=1}^{4} \mathbb{E}[I_i] = 4\left(\frac{3}{4}\right)^8.$$

4. Each Pokémon card is either rare or common. Also, any Pokémon card (either rare or common) can be holographic (holo) or normal. A Pokémon card is rare with probability 1/10. Given that a card is rare, it is holo with probability 1/5. Given that a card is common, it is holo with probability 1/7. Each pack of Pokémon cards contains 10 cards, and the properties of all cards (in the same pack or different packs) are mutually independent and follow the model above.

(a) Given that a card is holo, what is the conditional probability that it is rare?

Answer. Let R be the event that the card is rare (so R' is the event that it is common), and let H be the event that the card is holo. We are given

$$P(R) = \frac{1}{10}, \qquad P(H \mid R) = \frac{1}{5}, \qquad P(H \mid R') = \frac{1}{7}.$$

By the law of total probability,

$$P(H) = P(H \mid R)P(R) + P(H \mid R')P(R') = \frac{1}{5} \cdot \frac{1}{10} + \frac{1}{7} \cdot \frac{9}{10} = \frac{1}{50} + \frac{9}{70}$$

By Bayes' theorem,

$$P(R \mid H) = \frac{P(H \mid R)P(R)}{P(H)} = \frac{1/50}{\frac{1}{50} + \frac{9}{70}}$$

(b) For a single pack, what is the expected number of common cards that are holo?

Answer. For a single card,

$$P(R' \cap H) = P(H \mid R')P(R') = \frac{1}{7} \cdot \frac{9}{10} = \frac{9}{70}.$$

Since the 10 cards in a pack are independent, the number of common holo cards in a pack has the Binom$(10, 9/70)$ distribution. Hence the expected number is

$$10 \cdot \frac{9}{70} = \frac{9}{7}.$$

(c) A pack is called *valuable* if it contains at least one rare card that is holo. You open packs until you open your fourth valuable pack (and then you stop). What is the probability that you have to open exactly 20 packs?

Answer. For a single card,

$$P(R \cap H) = P(H \mid R)P(R) = \frac{1}{5} \cdot \frac{1}{10} = \frac{1}{50}.$$

A pack fails to be valuable exactly when none of its 10 cards is rare and holo. By independence of the cards, this has probability $(49/50)^{10}$, so a pack is valuable with probability

$$p = 1 - \left(\frac{49}{50}\right)^{10}.$$

Since different packs are independent, the packs form a sequence of independent Bernoulli trials with success probability p. Let X be the number of packs opened. Then $X \sim \mathrm{NB}(4, p)$, so

$$P(X = 20) = \binom{19}{3} p^4 (1-p)^{16} = \binom{19}{3} \left(1 - \left(\frac{49}{50}\right)^{10}\right)^4 \left(\frac{49}{50}\right)^{160}.$$

5. Calls arrive at a help desk according to a Poisson process with rate 3 calls per hour. Time is measured in hours, starting at $t = 0$.

(a) What is the probability that the second call arrives after the first 3 hours?

Answer. The second call arrives after time 3 exactly when at most one call arrives in the first 3 hours, i.e., when $N[0,3] \le 1$. Since $N[0,3] \sim \mathrm{Pois}(9)$,

$$P(N[0,3] \le 1) = P(N[0,3] = 0) + P(N[0,3] = 1) = e^{-9} + 9e^{-9} = 10e^{-9}.$$

(b) Given that exactly 4 calls arrive in the first 2 hours, what is the conditional probability that all 4 of them arrive during the first hour?

Answer. Note that $N[0,1] \sim \mathrm{Pois}(3)$, $N(1,2] \sim \mathrm{Pois}(3)$, $N[0,2] \sim \mathrm{Pois}(6)$, and $N[0,1]$, $N(1,2]$ are independent. Hence

$$P\big(N[0,1] = 4 \,\big|\, N[0,2] = 4\big) = \frac{P\big(N[0,1] = 4,\ N(1,2] = 0\big)}{P\big(N[0,2] = 4\big)}$$

$$= \frac{P\big(N[0,1] = 4\big)P\big(N(1,2] = 0\big)}{P\big(N[0,2] = 4\big)} = \frac{e^{-3}\frac{3^4}{4!} \cdot e^{-3}}{e^{-6}\frac{6^4}{4!}} = \frac{3^4}{6^4} = \frac{1}{16}.$$

(c) Given that exactly 4 calls arrive in the first 2 hours, what is the conditional probability that the second call arrived during the first hour?

Answer. The second call arrives during the first hour exactly when $N[0,1] \geq 2$. For $k = 0, 1, 2, 3, 4$, the same computation as in part (b) gives

$$P\big(N[0,1] = k \,\big|\, N[0,2] = 4\big) = \frac{P\big(N[0,1] = k,\ N(1,2] = 4-k\big)}{P\big(N[0,2] = 4\big)}$$

$$= \frac{e^{-3}\frac{3^k}{k!} \cdot e^{-3}\frac{3^{4-k}}{(4-k)!}}{e^{-6}\frac{6^4}{4!}} = \binom{4}{k}\left(\frac{1}{2}\right)^4.$$

Therefore,

$$P\big(N[0,1] \geq 2 \,\big|\, N[0,2] = 4\big) = 1 - \binom{4}{0}\left(\frac{1}{2}\right)^4 - \binom{4}{1}\left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16} - \frac{4}{16} = \frac{11}{16}.$$