On Gaps in the Closures of Images of Divisor Functions

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Divisor Functions

Definition

Fix some $c \in \mathbb{C}$. The **divisor function** $\sigma_c : \mathbb{N} \to \mathbb{C}$ associated to c is given by

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Example

Fix a prime p and pick some $\alpha \in \mathbb{N}$. Then,

$$\sigma_c(p) = 1 + p^c$$

$$\sigma_c(p^\alpha) = 1 + p^c + \cdots + p^{\alpha c} = \frac{1 - p^{(\alpha+1)c}}{1 - p^c}.$$

The Question

Theorem (Laatsch, 1986)

$\sigma_{-1}(\mathbb{N})$ *is a dense subset of* $[1, \infty)$. *That is,*

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Question

Is $\overline{\sigma_c(\mathbb{N})}$ *always connected?*

The Answer (and a new Question)

Answer: No

Example

Consider $c = -2$. Then,

$$\begin{aligned} n \text{ even} \quad &\Longrightarrow \quad \sigma_{-2}(n) \;\geq\; 1 + \tfrac{1}{2^2} \;=\; \tfrac{5}{4} \\ n \text{ odd} \quad &\Longrightarrow \quad \sigma_{-2}(n) \;\leq\; \sum_{d \geq 1} \tfrac{1}{(2d-1)^2} \;=\; \tfrac{\pi^2}{8}. \end{aligned}$$

Hence, $\overline{\sigma_{-2}(\mathbb{N})}$ has a gap $\left(\frac{\pi^2}{8}, \frac{5}{4}\right)$.

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Hence, $\overline{\sigma_{-2}(\mathbb{N})}$ has a gap $\left(\frac{\pi^2}{8}, \frac{5}{4}\right)$.

Question

Fix some $r > 1$ and let $c = -r$. What can we say about the number of connected components of $\overline{\sigma_{-r}(\mathbb{N})}$?

Previous Results

Notation

Let C_r denote the number of connected components of $\overline{\sigma_{-r}(\mathbb{N})}$.

Theorem (Defant, 2015)

There is a number $\eta \approx 1.88779$*, now called the* ***Defantstant****, s.t.*

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Theorem (Sanna, 2017)

$C_r < \infty$ *for all* $r > 1$.

Known Results, Continued

Remark: We know that C_r is always finite and can be arbitrarily large. It seems reasonable to conjecture that it can take on all finite values.

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We say $m \in \mathbb{N}$ is a **Zubrilina number** if $C_r \neq m$ for all $r > 1$.

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Theorem (A., Berger)

6 is a Zubrilina number.

A Simplification

Definition (Steinitz, 1910)

A **Steinitz** (or **supernatural**) **number** is a formal product

$$n = \prod_{p \text{ prime}} p^{\alpha_p} \qquad \alpha_p \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

We write $v_p(n) = \alpha_p$, and we let $\mathbb{S}$ denote the set of all Steinitz numbers.

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Theorem (A., Berger)

Given $c \in \mathbb{C}$ with $\Re(c) < -1$, we can extend σ_c to a function $\mathbb{S} \to \mathbb{C}$ by preserving multiplicativity and setting

$$\sigma_c(p^\infty) = \lim_{n \to \infty} \sigma_c(p^n) = \frac{1}{1 - p^c}$$

When we do so, we get that $\overline{\sigma_c(\mathbb{N})} = \sigma_c(\mathbb{S})$.

Mighty Primes

Definition (Zubrilina, 2017)

Given $r > 1$, we say that a prime p is r**-mighty** if

$$\sigma_{-r}(p) = 1 + \frac{1}{p^r} > \prod_{\text{prime } q > p} \frac{1}{1 - q^{-r}} = \prod_{\text{prime } q > p} \sigma_{-r}(q^\infty).$$

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Example (2 is 2-mighty)

For $r = 2$, we have that

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Notation

- Let p_k denote the kth prime.
- Let P_r denote the smallest prime larger than all r-mighty primes.

r-mighty Primes $\Longrightarrow$ Gaps

Theorem (Zubrilina, 2017)

If p_k is an r-mighty prime, then

$$\left(\prod_{t=k+1}^{\infty} \sigma_{-r}(p_t^{\infty}), \sigma_{-r}(p_k)\right)$$

is a gap of $\sigma_{-r}(\mathbb{S})$.

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Theorem (A., Berger)

If q, p_k are r-mighty primes with $q^2 < p_k$, then

$$\left(\sigma_{-r}(q) \prod_{t=k+1}^{\infty} \sigma_{-r}(p_t^{\infty}), \sigma_{-r}(qp_k)\right)$$

is a gap of $\sigma_{-r}(\mathbb{S})$.

Gaps Coming From Other Places

Lemma

If $1 \leq d \leq r - 1$, *then*

$$\frac{1}{d^r} > \sum_{n=d+1}^{\infty} \frac{1}{n^r}.$$

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If n *is* $(r-1)$*-smooth and square-free, then* $\sigma_{-r}(n)$ *is the right endpoint of a gap of* $\sigma_{-r}(\mathbb{S})$*.*

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Corollary

For $r > 1$, *we have* $C_r \geq 2^{\pi(r-1)}$.

Structure of General Gaps

Theorem (A., Berger)

Let $(\sigma_{-r}(a), \sigma_{-r}(b))$ *with* $a, b \in \mathbb{S}$ *be a gap of* $\sigma_{-r}(\mathbb{S})$*. Then,*

- *If* $p \geq P_r$*, then*

$$v_p(a) = \infty \text{ and } v_p(b) = 0.$$

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- *If* $p < P_r$ *and* $v_p(a) < \infty$, *then*
$$v_p(a) < \frac{\log P_r}{\log p} - 1.$$

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$$v_p(b) < \frac{\log P_r}{\log p}.$$

Remark: The above theorem implies that b must be a natural number.

Open Problems

Conjecture (Defant, 2018)

There are infinitely many Zubrilina numbers.

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C_r *is monotone in* r.

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