An upper bound for the average rank of elliptic curves over arbitrary (global) function fields

Niven Achenjang
MIT

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Defining average ranks

Notation throughout talk:

- B = nice $\mathbb{F}_q$-curve of genus g.
- $K = \mathbb{F}_q(B)$ its function field.
- $\Delta_{\min}(E) \in \mathrm{Div}(B)$ is the minimal discriminant of an elliptic curve (EC) E/K.

Definition

The `height` of E/K is

$$\mathrm{ht}(E) := \frac{1}{12}\deg \Delta_{\min}(E),$$

essentially the degree of its minimal discriminant.

Definition

$$\mathbb{E}[\text{rank } E(K)] \overset{*}{:=} \limsup_{X\to\infty} \frac{\sum_{E:\ \mathrm{ht}(E)<X} \text{rank } E(K)}{\#\{E:\ \mathrm{ht}(E) < X\}}.$$

Conjectural distribution & Selmer groups

Conjecture (Minimalist conjecture, Goldfeld++)

50% of elliptic curves have rank 0 and 50% have rank 1. Thus,

$$\mathbb{E}[\operatorname{rank} E(K)] = \frac{1}{2}.$$

General plan of attack: first study Selmer group statistics.

Definition

Given E/K and $n \geq 1$, the n-`Selmer group` is

$$\mathrm{Sel}_n(E) := \ker\left(\mathrm{H}^1(K, E[n]) \longrightarrow \prod_v \mathrm{H}^1(K_v, E) \right).$$

Think: **locally solvable** genus 1 curve C equipped with degree n map $C \to \mathbb{P}^{n-1}$ (More on this later in the talk).

Selmer group statistics

Fact

$\text{Sel}_n(E)$ is finite and $E(K)/n \hookrightarrow \text{Sel}_n(E)$, so

$$n^{\text{rank}\, E(K)} \le \#\, \text{Sel}_n(E).$$

Definition

$$\mathbb{E}[\#\, \text{Sel}_n(E/K)] \overset{*}{:=} \limsup_{X \to \infty} \frac{\sum_{E:\ \text{ht}(E) < X} \#\, \text{Sel}_n(E)}{\#\{E:\ \text{ht}(E) < X\}}.$$

Conjecture (Bhargava–Shankar '13, Poonen–Rains '12, ...)

For every $n \ge 1$: $\mathbb{E}[\#\, \text{Sel}_n(E/K)] = \sum_{d \mid n} d$.

Fact: This conjecture $\implies \mathbb{P}[\text{rank}\, E(K) \le 1] = 100\%$.
Combined with equidistribution of rank parities, this implies the minimalist conjecture.

Results for 2-Selmer Over Function Fields

Theorem (Hồ–Lê Hùng–Ngô '14)

Assume that $\operatorname{char} \mathbb{F}_q \neq 2, 3$ *and that* $q > 64$. *Then,*

$$3\zeta_B(10)^{-1} \leq \mathbb{E}[\# \operatorname{Sel}_2(E/K)] \leq 3 + O_g\left(\frac{1}{q}\right),$$

as $q \to \infty$, *where* ζ_B *is the usual zeta function of* B.

Theorem (A. '23)

$$\#\mathbb{E}[\operatorname{Sel}_2(E/K)] \leq 1 + 2\zeta_B(2)\zeta_B(10) = 3 + \frac{2}{q} + O_g\left(\frac{1}{q^2}\right),$$

as $q \to \infty$.

In either case, $\mathbb{E}[\operatorname{rank} E(K)] \leq 3/2 + O_g(1/q)$. Use $2x \leq 2^x$.

Corollary (A. '23 + Shankar's thesis)

$\mathbb{E}[\operatorname{rank} E(F)] < \infty$ *for* ***any global field*** F.

2-Selmer elements as double covers of $\mathbb{P}^1$

Goal: Count 2-Selmer elements of ECs of bounded height.

Remark (geometric model of 2-Selmer elements)

If $T \in \mathrm{H}^1(K, E[2])$ is an $E[2]$-torsor, can twist the diagram $E \xrightarrow{x} \mathbb{P}^1$ by T to obtain

$$\underbrace{C}_{E\text{-torsor}} \longrightarrow \underbrace{S}_{\text{genus } 0 \text{ curve}} .$$

If C is locally solvable, then $S \simeq \mathbb{P}^1_K$.

Upshot: $\alpha \in \mathrm{Sel}_2(E) \subset \mathrm{H}^1(K, E[2])$ represented by a locally solvable genus 1 double cover $C \to \mathbb{P}^1_K$ with $E \cong \mathrm{Jac}(C)$. Such C given by equation

$$C \colon Y^2 + (a_0X^2 + a_1XZ + a_2Z^2)Y = c_0X^4 + c_1X^3Z + c_2X^2Z^2 + c_3XZ^3 + c_4Z^4$$

inside $\mathbb{P}(1,2,1)$. Above equation has 8 coefficients.

Setup to counting (i.e. relativizing everything), I

Goal: Count (locally solvable) C's as below with Jacobians of bounded height.

$$C\colon Y^2 + (a_0X^2 + a_1XZ + a_2Z^2)Y = c_0X^4 + c_1X^3Z + c_2X^2Z^2 + c_3XZ^3 + c_4Z^4$$

Recall **height** measured bad reduction of $E = \text{Jac}(C)$, so easiest to read off an integral (i.e. relative over B) model.

Fact

There exists a rank 2 vector bundle $\mathscr{E} = \mathscr{E}(C)$ on B and an integral model

$$\mathcal{C} \to \mathbb{P}(\mathscr{E}) \text{ of } C \to \mathbb{P}^1$$

such that $\text{ht}(\mathcal{C}) = \text{ht}(E)$. *Furthermore, one can embed*

$$\mathcal{C} \hookrightarrow \mathbb{P}$$

for some $\mathbb{P}(1,2,1)$*-bundle* $\mathbb{P}$ *over* B.

Setup to counting (i.e. relativizing everything), II

Goal: Count diagrams

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathbb{P} \\ & \searrow \quad \swarrow & \\ & B, & \end{array}$$

where $\mathbb{P}/B$ is a $\mathbb{P}(1,2,1)$-bundle and $\mathcal{C}/B$ is a relative curve whose generic fiber looks like C from before.

Fact

If one fixes $\mathbb{P}/B$, there is a rank 8 vector bundle $\mathscr{V} = \mathscr{V}(\mathbb{P})$ over B whose sections cut out $\mathcal{C}$'s as above (its sections parameterize continuous choices of the 8 coefficients $a_0, a_1, a_2, c_0, c_1, c_2, c_3, c_4$ from before).

Upshot: One ultimately wants to count $\mathcal{C}$'s by computing $h^0(\mathscr{V})$.

Computing $h^0(\mathscr{V})$, I

Fix $\mathbb{P}/B$, a $\mathbb{P}(1,2,1)$-bundle. Goal: Compute $h^0(\mathscr{V})$.

Fact

$\chi(\mathscr{V}) = \deg \mathscr{V} + 8(1-g)$ (Riemann-Roch) is easily computable.

Note: If $h^1(\mathscr{V}) \approx 0$, then $h^0(\mathscr{V}) \approx \chi(\mathscr{V})$.

Fact

Attached to $\mathbb{P}$ is a rank 2 vector bundle $\mathscr{E}$ such that $\mathscr{V}$ is "basically built from"

$$\mathrm{Sym}^2(\mathscr{E}) \ \textit{and} \ \mathrm{Sym}^4(\mathscr{E})$$

(corresponding to binary quadratic form and binary quartic form in previous equation for C)

Computing $h^0(\mathcal{V})$, II

Fix $\mathbb{P}/B$, a $\mathbb{P}(1,2,1)$-bundle. Goal: Show $h^1(\mathscr{V}) \approx 0$.

Fact ("doing enough AG let's you better understand $\mathscr{E}$")

Say $\mathcal{C}$ has 'at worst rational singularities' (= 'is minimal'). One can compute $\mathscr{E}$ using data on a desingularization $\mathcal{X}$ of $\mathcal{C}$. Some intersection theory on $\mathcal{X}$ allows one to show that $\mathscr{E}$ is "almost semistable," unless $\mathcal{C}$ corresponds to a trivial/identity Selmer element.

Output: One uses that $\mathscr{V}$ is "built from $\mathrm{Sym}^2(\mathscr{E})$ and $\mathrm{Sym}^4(\mathscr{E})$" to ultimately show that

$$h^1(\mathscr{V}) \leq 8g \text{ and so } h^0(\mathscr{V}) \approx \chi(\mathscr{V}),$$

at least for $\mathbb{P}$'s (so also $\mathscr{V}$'s) attached to non-trivial Selmer elements. This let's one count ("non-trivial") $\mathcal{C}$'s, so counts C's, and so count 2-Selmer elements.

Some hiccups to watch out for

- One also needs to compute the denominator, i.e. count elliptic curves $/K$.

Theorem (A. '23)

For any $\varepsilon > 0$,

$$\sum_{E:\ \mathrm{ht}(E)=d} \frac{1}{\#\operatorname{Aut}(E)} \sim \#\operatorname{Pic}^0(B)\frac{q^{10d+2(1-g)}}{(q-1)\zeta_B(10)} \ \textit{as}\ d \to \infty$$

(one also gets an explicit second order term).

- Elliptic curves with non-trivial 2-torsion turn out to appear differently in the Selmer count, so one wants to show they don't contribute to the average anyways.
  - This is not too bad if char $K \geq 3$, but turns out to be more subtle if char $K = 2$.

Summary

- We bounded the average rank of elliptic curves over arbitrary global function fields (including small characteristic).
- This completed the proof that this average rank is finite for any global field.
- The proof involved first bounding the average size of 2-Selmer, primarily by relating 2-Selmer elements to certain integral models of double covers of $\mathbb{P}^1$.
- Such integral models can be counted by using tools from algebraic geometry, e.g. facts about vector bundles on curves and the theory of rational singularities.

Thank you!