# An upper bound for the average rank of elliptic curves over arbitrary (global) function fields

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 $\mathbb{E}[\operatorname{rank} E(K)] < \infty$ 

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

## Defining average ranks

Notation throughout talk:

- $B = \text{nice } \mathbb{F}_q$ -curve of genus g.
- $K = \mathbb{F}_q(B)$  its function field.
- ∆<sub>min</sub>(E) ∈ Div(B) is the minimal discriminant of an elliptic curve (EC) E/K.

#### Definition

The height of E/K is

$$\operatorname{nt}(E)\coloneqq rac{1}{12}\deg\Delta_{\min}(E),$$

essentially the degree of its minimal discriminant.

#### Definition

$$\mathbb{E}[\mathsf{rank}\, E(\mathcal{K})] \stackrel{*}{:=} \limsup_{X \to \infty} \frac{\sum\limits_{E: \ \mathsf{ht}(E) < X} \mathsf{rank}\, E(\mathcal{K})}{\#\{E: \ \mathsf{ht}(E) < X\}}.$$

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Parameterizing 2-Selmer Elements

Counting

## Conjectural distribution & Selmer groups

Conjecture (Minimalist conjecture, Goldfeld++) 50% of elliptic curves have rank 0 and 50% have rank 1. Thus,  $\mathbb{E}[\operatorname{rank} E(K)] = \frac{1}{2}.$ 

General plan of attack: first study Selmer group statistics.

#### Definition

Given 
$$E/K$$
 and  $n \ge 1$ , the *n*-Selmer group is

$$\operatorname{Sel}_n(E) \coloneqq \operatorname{ker}\left(\operatorname{H}^1(K, E[n]) \longrightarrow \prod_{v} \operatorname{H}^1(K_v, E)\right).$$

<u>Think</u>: **locally solvable** genus 1 curve *C* equipped with degree  $n \text{ map } C \to \mathbb{P}^{n-1}$  (More on this later in the talk).

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

## Selmer group statistics

Fact

$$\operatorname{Sel}_n(E)$$
 is finite and  $E(K)/n \hookrightarrow \operatorname{Sel}_n(E)$ , so  
 $n^{\operatorname{rank} E(K)} \leq \# \operatorname{Sel}_n(E).$ 

Definition

$$\mathbb{E}[\#\operatorname{Sel}_n(E/K)] \stackrel{*}{:=} \limsup_{X \to \infty} \frac{\sum\limits_{E: \ \operatorname{ht}(E) < X} \#\operatorname{Sel}_n(E)}{\#\{E: \ \operatorname{ht}(E) < X\}}.$$

Conjecture (Bhargava–Shankar '13, Poonen–Rains '12, ...) For every  $n \ge 1$ :  $\mathbb{E}[\# \operatorname{Sel}_n(E/K)] = \sum_{d|n} d$ .

<u>Fact</u>: This conjecture  $\implies \mathbb{P}[\operatorname{rank} E(K) \le 1] = 100\%$ . Combined with equidistribution of rank parities, this implies the minimalist conjecture.

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

## Results for 2-Selmer Over Function Fields

Theorem (Hồ–Lê Hùng–Ngô '14)

Assume that char  $\mathbb{F}_q \neq 2,3$  and that q > 64. Then,

$$3\zeta_B(10)^{-1} \leq \mathbb{E}[\#\operatorname{Sel}_2(E/K)] \leq 3 + O_g\left(rac{1}{q}
ight),$$

as  $q \to \infty$ , where  $\zeta_B$  is the usual zeta function of B.

Theorem (A. '23)

a

$$\#\mathbb{E}[\operatorname{Sel}_2(E/\mathcal{K})] \leq 1 + 2\zeta_B(2)\zeta_B(10) = 3 + rac{2}{q} + O_g\left(rac{1}{q^2}
ight),$$
  
is  $q o \infty.$ 

In either case,  $\mathbb{E}[\operatorname{rank} E(K)] \leq 3/2 + O_g(1/q)$ . Use  $2x \leq 2^x$ .

Corollary (A. '23 + Shankar's thesis)  $\mathbb{E}[\operatorname{rank} E(F)] < \infty$  for any global field F.  $\mathbb{E}[\operatorname{rank} E(K)] <$ 

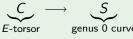
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## 2-Selmer elements as double covers of $\mathbb{P}^1$

Goal: Count 2-Selmer elements of ECs of bounded height.

Remark (geometric model of 2-Selmer elements)

If  $T \in H^1(K, E[2])$  is an E[2]-torsor, can twist the diagram  $E \xrightarrow{x} \mathbb{P}^1$  by T to obtain



If *C* is locally solvable, then  $S \simeq \mathbb{P}^1_K$ .

<u>Upshot</u>:  $\alpha \in Sel_2(E) \subset H^1(K, E[2])$  represented by a locally solvable genus 1 double cover  $C \to \mathbb{P}^1_K$  with  $E \cong Jac(C)$ . Such *C* given by equation

$$C: Y^{2} + (a_{0}X^{2} + a_{1}XZ + a_{2}Z^{2})Y = c_{0}X^{4} + c_{1}X^{3}Z + c_{2}X^{2}Z^{2} + c_{3}XZ^{3} + c_{4}Z^{4}$$

inside  $\mathbb{P}(1,2,1)$ . Above equation has 8 coefficients.

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

Nrapup

## Setup to counting (i.e. relativizing everything), I

<u>Goal</u>: Count (locally solvable) *C*'s as below with Jacobians of bounded height.

C: 
$$Y^2 + (a_0X^2 + a_1XZ + a_2Z^2)Y =$$
  
 $c_0X^4 + c_1X^3Z + c_2X^2Z^2 + c_3XZ^3 + c_4Z^4$ 

Recall **height** measured bad reduction of E = Jac(C), so easiest to read off an integral (i.e. relative over B) model.

#### Fact

There exists a rank 2 vector bundle  $\mathscr{E} = \mathscr{E}(C)$  on B and an integral model

$$\mathcal{C} \to \mathbb{P}(\mathscr{E})$$
 of  $\mathcal{C} \to \mathbb{P}^1$ 

such that  $ht(\mathcal{C}) = ht(E)$ . Furthermore, one can embed

$$\mathfrak{C} \hookrightarrow \mathbb{P}$$

for some  $\mathbb{P}(1,2,1)$ -bundle  $\mathbb{P}$  over B.

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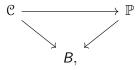
Background & Setup

Parameterizing 2-Selmer Elements

Counting

## Setup to counting (i.e. relativizing everything), II

Goal: Count diagrams



where  $\mathbb{P}/B$  is a  $\mathbb{P}(1,2,1)$ -bundle and  $\mathbb{C}/B$  is a relative curve whose generic fiber looks like *C* from before.

#### Fact

If one fixes  $\mathbb{P}/B$ , there is a rank 8 vector bundle  $\mathscr{V} = \mathscr{V}(\mathbb{P})$ over B whose sections cut out C's as above (its sections parameterize continuous choices of the 8 coefficients  $a_0, a_1, a_2, c_0, c_1, c_2, c_3, c_4$  from before).

Upshot: One ultimately wants to count  $\mathcal{C}$ 's by computing  $h^0(\mathcal{V})$ .

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

## Computing $h^0(\mathscr{V})$ , I

Fix  $\mathbb{P}/B$ , a  $\mathbb{P}(1,2,1)$ -bundle. <u>Goal</u>: Compute  $h^0(\mathcal{V})$ .

#### Fact

 $\chi(\mathscr{V}) = \deg \mathscr{V} + 8(1 - g)$  (Riemann-Roch) is easily computable.

Note: If 
$$h^1(\mathscr{V}) \approx 0$$
, then  $h^0(\mathscr{V}) \approx \chi(\mathscr{V})$ .

#### Fact

Attached to  $\mathbb{P}$  is a rank 2 vector bundle  $\mathscr{E}$  such that  $\mathscr{V}$  is "basically built from"

 $\operatorname{Sym}^2(\mathscr{E})$  and  $\operatorname{Sym}^4(\mathscr{E})$ 

(corresponding to binary quadratic form and binary quartic form in previous equation for C)

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

Nrapup

## Computing $h^0(\mathcal{V})$ , II

Fix  $\mathbb{P}/B$ , a  $\mathbb{P}(1,2,1)$ -bundle. <u>Goal</u>: Show  $h^1(\mathscr{V}) \approx 0$ .

Fact ("doing enough AG let's you better understand  $\mathscr{E}$ ") Say C has 'at worst rational singularities' (= 'is minimal'). One can compute  $\mathscr{E}$  using data on a desingularization  $\mathfrak{X}$  of C. Some intersection theory on  $\mathfrak{X}$  allows one to show that  $\mathscr{E}$  is "almost semistable," unless C corresponds to a trivial/identity Selmer element.

Output: One uses that  $\mathscr{V}$  is "built from  $Sym^2(\mathscr{E})$  and  $\overline{Sym^4}(\mathscr{E})$ " to ultimately show that

 $h^1(\mathscr{V}) \leq 8g$  and so  $h^0(\mathscr{V}) \approx \chi(\mathscr{V}),$ 

at least for  $\mathbb{P}$ 's (so also  $\mathscr{V}$ 's) attached to non-trivial Selmer elements. This let's one count ("non-trivial")  $\mathcal{C}$ 's, so counts C's, and so count 2-Selmer elements.

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Background & Setup

Parameterizing 2-Selmer Elements

Counting

### Some hiccups to watch out for

One also needs to compute the denominator, i.e. count elliptic curves /K.

Theorem (A. '23) For any  $\varepsilon > 0$ ,  $\sum_{E: ht(E)=d} \frac{1}{\#\operatorname{Aut}(E)} \sim \#\operatorname{Pic}^0(B) \frac{q^{10d+2(1-g)}}{(q-1)\zeta_B(10)} \text{ as } d \to \infty$ Wrapup (one also gets an explicit second order term).

- Elliptic curves with non-trivial 2-torsion turn out to appear differently in the Selmer count, so one wants to show they don't contribute to the average anyways.
  - This is not too bad if char  $K \ge 3$ , but turns out to be more subtle if char K = 2.

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## Summary

- We bounded the average rank of elliptic curves over arbitrary global function fields (including small characteristic).
- This completed the proof that this average rank is finite for any global field.
- ► The proof involved first bounding the average size of 2-Selmer, primarily by relating 2-Selmer elements to certain integral models of double covers of P<sup>1</sup>.
- Such integral models can be counted by using tools from algebraic geometry, e.g. facts about vector bundles on curves and the theory of rational singularities.

Thank you!

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Background & Setup

Parameterizing 2-Selmer Elements

Counting