MODULARITY SEMINAR: TAYLOR-WILES DEFORMATIONS

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Abstract. We will closely follow [Gee22, §3], and go through some explicit computations of local deformation rings, in the setting $\ell \neq p$ (i.e., p-adic representations of ℓ-adic Galois groups).

Let $p \neq \ell$, let $K/\mathbb{Q}_\ell$ be a finite extension, and let $L/\mathbb{Q}_p$ be an algebraic extension. Suppose $\overline{\rho}\colon G_K \to \mathrm{GL}_n(k_L)$ is a representation, and let χ be a character $G_K \to \mathcal{O}_L^\times$, i.e., a character $G_K^{\mathrm{ab}} \simeq \widehat{K^\times}$. Recall that our goal is to characterize $R^\square_{\overline{\rho},\chi}$, which we recall is the representing object of the functor

$$\mathcal{R}^\square_{\overline{\rho},\chi}\colon \mathcal{C}_{\mathcal{O}_L} \to \mathbf{Sets}$$
$$(A, \mathfrak{m}_A) \mapsto \left\{\begin{matrix}\text{continuous representations } \rho\colon G_K \to \mathrm{GL}_2(A) \\ \text{such that } \overline{\rho} = \rho \bmod \mathfrak{m}_A \text{ and } \det(\rho) = \chi\end{matrix}\right\}.$$

The full deformation ring has several irreducible components, and to extract each component, we control what the p-adic representation $G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ looks like, for each homomorphism $A \to \overline{\mathbb{Q}}_p$. Each Galois representation $G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ can be described in a more combinatorial way, a *Weil-Deligne representation*, by Grothendieck's monodromy theorem.

More precisely, Grothendieck's monodromy theorem defines a map

$$\{\text{continuous representations } G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)\} \to \{\text{inertial WD-type}\},$$

and for each inertial WD-type τ we consider the deformation problem

$$\mathcal{R}^\square_{\overline{\rho},\chi,\tau}\colon \mathcal{C}_{\mathcal{O}_L} \to \mathbf{Sets}$$
$$(A, \mathfrak{m}_A) \mapsto \left\{\begin{matrix}\text{continuous representations } \rho\colon G_K \to \mathrm{GL}_2(A) \text{ such that } \overline{\rho} = \rho \bmod \mathfrak{m}_A \\ \text{and } \det(\rho) = \chi, \text{ and for all } \psi\colon A \to \overline{\mathbb{Q}}_p,\ \psi \circ \rho \text{ has inertial WD-type } \tau\end{matrix}\right\}.$$

These functors turn out to be representable closed sub-functors, and give rise to all components of $\mathcal{R}^\square_{\overline{\rho},\chi}$.

1. Grothendieck's monodromy theorem

Let $\ell \neq p$ be two primes. Let $K/\mathbb{Q}_p$ be a finite extension, with residue field of size q_K. We will consider ℓ-adic representations of G_K, i.e., a representation into a finite-dimensional L-vector space, where $L/\mathbb{Q}_\ell$ is algebraic.

Definition 1.0.1. Let W_K be the Weil group of K. A *Weil-Deligne representation* of W_K on a finite-dimensional L-vector space V is a pair (r, N) where

$r\colon W_K \to \mathrm{GL}(V)$ is a continuous semisimple representation, and $N\colon V \to V$ is an endomorphism, such that for all $\sigma \in W_K$,

$$r(\sigma)Nr(\sigma)^{-1} = q_K^{-v_K(\sigma)}N.$$

A Weil-Deligne representation is *bounded* if for all $\sigma \in W_K$ the operator $r(\sigma)$ is bounded, i.e., the determinant is in $\mathcal{O}_L^\times$ and the characteristic polynomial is in $\mathcal{O}_L[X]$ (equivalently, all of the eigenvalues are in $\mathcal{O}_{\overline{L}}^\times$).

Now recall Grothendieck's monodromy theorem ([Gee22, Prop 2.18], [BH06, Thm 32.5], [ST68]):

Proposition 1.0.2. *Suppose $\ell \neq p$, let $K/\mathbb{Q}_\ell$ be a finite extension, let $L/\mathbb{Q}_p$ be an algebraic extension, and let V be a finite-dimensional L-vector space. Fix:*

- *φ, a lift of Fr_K; and*
- *a compatible system $(\zeta_m)_{(m,\ell)=1}$ of primitive roots of unity.*

Then for any continuous representation $\rho\colon G_K \to \mathrm{GL}(V)$ there is a finite extension K'/K and a uniquely determined nilpotent endomorphism $N\colon V \to V$ such that for all $\sigma \in I_{K'}$,

$$\rho(\sigma) = \exp(Nt_{\zeta,p}(\sigma)),$$

where for all $\sigma \in W_K$, we have $\rho(\sigma)N\rho(\sigma)^{-1} = q_K^{-v_K(\sigma)}N$, where t_ζ is an isomorphism $I_K/P_K \simeq \prod_{p\neq\ell}\mathbb{Z}_p$.

Moreover, there is an equivalence of categories:

$$\left\{\begin{matrix}\textit{continuous representations of } G_K \textit{ on}\\ \textit{finite-dimensional } L\textit{-vector spaces}\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{bounded Weil-Deligne representations}\\ \textit{on finite dimensional } L\textit{-vector spaces}\end{matrix}\right\}$$
$$\rho \mapsto (V, r, N),$$

where $r(\tau) := \rho(\tau)\exp(-t_{\zeta,p}(\varphi^{-v_K(\tau)}\tau)N)$.

Grothendieck's theorem allows us to define the *inertial WD-type* of a representation $\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$:

Definition 1.0.3. Let $\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation, and let (r, N) be the associated Weil-Deligne representation. The *inertial WD-type* of ρ is $(r|_{I_F}, N)$.

Now, fix a $\overline{\rho}\colon G_K \to \mathrm{GL}_2(k_L)$. Then we have the following general result on $R^\square_{\overline{\rho},\chi}$ [Gee22, Thm 3.31]:

Theorem 1.0.4. *$R^\square_{\overline{\rho},\chi}$ is equidimensional of Krull dimension 4, and the generic fiber $R^\square_{\overline{\rho},\chi}$ has Krull dimension 3. Furthermore:*

1. *(a) The function which takes a $\mathbb{Q}_p$-points $x\colon R^\square_{\overline{\rho},\chi}[1/p] \to \overline{\mathbb{Q}}_p$ to $WD(x \circ \rho^\square)|_{I_K}$ (forgetting N) is constant on the irreducible components of $R^\square_{\overline{\rho},\chi}[1/p]$*
2. *(b) The irreducible components of $R^\square_{\overline{\rho},\chi}[1/p]$ are all regular, and there are only finitely many of them.*

Now, we can define the deformation ring with fixed inertial WD type:

Proposition-Definition 1. Let τ be an inertial WD-type. Then $R^{\square}_{\overline{\rho},\chi}$ has a unique reduced p-torsion free quotient $R^{\square}_{\overline{\rho},\chi,\tau}$ such that for a continuous homomorphism $\psi\colon R^{\square}_{\overline{\rho},\chi} \to \overline{\mathbb{Q}}_p$, i.e., a Galois representation $\psi \circ \rho^{\square}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, the following are equivalent:

- $\psi \circ \rho^{\square}$ has inertial WD-type τ
- the homomorphism ψ factors through $R^{\square}_{\overline{\rho},\chi,\tau}$.

In other words,

$$\mathcal{R}^{\square}_{\overline{\rho},\chi,\tau}\colon \mathcal{C}_{\mathcal{O}_L} \to \mathbf{Sets}$$
$$(A,\mathfrak{m}_A) \mapsto \left\{\begin{matrix}\text{continuous representations } \rho\colon G_K \to \mathrm{GL}_2(A) \text{ such that } \overline{\rho} = \rho \bmod \mathfrak{m}_A \\ \text{and } \det(\rho) = \chi, \text{ and for all } \psi\colon A \to \overline{\mathbb{Q}}_p,\ \psi\circ\rho \text{ has inertial WD-type } \tau\end{matrix}\right\}.$$

is a closed sub-functor of $\mathcal{R}^{\square}_{\overline{\rho},\chi}$. When $R^{\square}_{\overline{\rho},\chi,\tau}$ is nonzero it has Krull dimension 4.

Proof. We will prove that $\mathcal{R}^{\square}_{\overline{\rho},\chi,\tau}$ is a closed sub-functor. It suffices to check the following: let $(A,\mathfrak{m}_A) \in \mathcal{C}_{\mathcal{O}_L}$ be geometrically irreducible, and let $\psi,\psi'\colon A \to \overline{\mathbb{Q}}_p$ be two homomorphisms. Then, $\psi\circ\rho$ and $\psi'\circ\rho$ have isomorphic restrictions to I_F (hence have the same inertial WD-type).

Now, the restriction of ρ to I_F factors through some finite group H, say of order m. Then let $E := A[1/p] \otimes_L L(\zeta_m)$, which is still an integral domain. Now we see that $\psi\circ\rho$ and $\psi'\circ\rho$ have the same trace, hence are isomorphic as H-representations. $\square$

In the following sections, we will go through some particular examples of these deformation rings.

2. Taylor-Wiles deformations

Now, suppose $\overline{\rho}\colon G_K \to \mathrm{GL}_n(k_L)$ is unramified, that $\overline{\rho}(\mathrm{Fr}_K)$ has distinct eigenvalues in k_L, that $q_K \equiv 1 \pmod p$, and let χ be an unramified character $G_K \to \mathcal{O}_L^{\times}$, i.e., a character of $G_K^{\mathrm{ab}} \simeq \widehat{K^{\times}}$. Our goal is to characterize $R^{\square}_{\overline{\rho},\chi}$, which we recall is the representing object of the functor

$$\mathcal{R}^{\square}_{\overline{\rho},\chi}\colon \mathcal{C}_{\mathcal{O}_L} \to \mathbf{Sets}$$
$$(A,\mathfrak{m}_A) \mapsto \left\{\begin{matrix}\text{continuous representations } \rho\colon G_K \to \mathrm{GL}_2(A) \\ \text{such that } \overline{\rho} = \rho \bmod \mathfrak{m}_A \text{ and } \det(\rho) = \chi\end{matrix}\right\}.$$

The following is [Gee22, Lemma 3.33]:

Lemma 2.0.1. *Let $q_K - 1$ be exactly divisible by p^m, with $m > 0$. Then*

$$R^{\square}_{\overline{\rho},\chi} \simeq \mathcal{O}_L[[x,y,a,s]]/((1+s)^{p^m} - 1).$$

Furthermore, if $\varphi \in G_K$ is a lift of Fr_K, then $\rho^{\square}(\varphi)$ is conjugate to a diagonal matrix.

Proof. First of all, $\rho^{\square}$ is tamely ramified, i.e., $\rho^{\square}(P_K) = \{1\}$, since $\rho^{\square}(P_K)$ is a pro-ℓ-subgroup of the pro-p-group $\ker(\mathrm{GL}_2(R^{\square}_{\overline{\rho},\chi}) \to \mathrm{GL}_2(k_L))$. Now let $\varphi \in G_K/P_K$

be a fixed lift of Fr_K, and let σ be a topological generator of I_K/P_K, which can be chosen so

$$\varphi^{-1}\sigma\varphi = \sigma^{q_K}.$$

Remark 2.0.2. The importance of φ and ρ come from the following: G_K/P_K is topologically generated by φ and ρ, with the only relation $\varphi^{-1}\sigma\varphi = \sigma^{q_K}$.

Write

$$\overline{\rho}(\varphi) = \begin{pmatrix} \overline{\alpha} & \\ & \overline{\beta} \end{pmatrix}$$

for $\alpha, \beta \in k_K$.

Now, let $(A, \mathfrak{m}_A) \in \mathcal{C}_{\mathcal{O}_L}$ and let $\rho\colon G_K \to \mathrm{GL}_2(A)$ be a lift of $\overline{\rho}$. Then by Hensel's lemma, there are $a, b \in \mathfrak{m}_A$ such that $\rho(\varphi)$ has characteristic polynomial $(X - (\alpha + a))(X - (\beta + b))$, i.e., $\rho(\varphi)$ has eigenvalues $\alpha + a$ and $\beta + b$. Since the determinant is $\chi(\varphi)$, we have $\beta + b = \chi(\varphi)/(\alpha + a)$. Moreover, the eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$ of $\overline{\rho}(\varphi)$ lift to eigenvectors:

$$\rho(\varphi)\begin{pmatrix} 1 \\ x \end{pmatrix} = (\alpha + a)\begin{pmatrix} 1 \\ x \end{pmatrix}$$
$$\rho(\varphi)\begin{pmatrix} y \\ 1 \end{pmatrix} = (\beta + b)\begin{pmatrix} y \\ 1 \end{pmatrix},$$

where $x, y \in \mathfrak{m}_A$.

Let ρ' be ρ but with a change of basis, i.e., by the conjugation of ρ by $\begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}$. Thus $\rho'(\varphi) = \begin{pmatrix} \alpha + a & \\ & \beta + b \end{pmatrix}$. Now, since $\overline{\rho}(\varphi) = 1$ since $\overline{\rho}$ is unramified, so there are $s, t, u, v \in \mathfrak{m}_A$ such that

$$\rho'(\sigma) = \begin{pmatrix} 1 + s & t \\ u & 1 + v \end{pmatrix}.$$

Since

$$\rho'(\varphi)^{-1}\rho'(\sigma)\rho'(\varphi) = \rho'(\sigma)^{q_K}$$

is a diagonal matrix, we see $t = u = 0$. Moreover, since the determinant of $\rho'(\sigma)$, which is $\chi(\sigma)$, is 1, we have $(1 + s)(1 + v) = 1$.

The commutator relation further implies that $(1 + s)^{q_K} = 1 + s$. Since $1 + s$ is invertible, we see that $(1 + s)^{q_K - 1} = 1$. Now recall that $q_K - 1 = p^m j$ where j is coprime to p. Since $1 + s \in 1 + \mathfrak{m}_A$ where $1 + \mathfrak{m}_A$ is a pro-p group, the j-th power map is invertible, and hence $(1 + s)^{p^m} = 1$.

All the above arguments have produced a bijection:

$$\mathcal{R}^{\square}_{\overline{\rho},\chi}(A, \mathfrak{m}_A) := \left\{ \begin{matrix} \text{continuous representations } \rho\colon G_K \to \mathrm{GL}_2(A) \\ \text{such that } \overline{\rho} = \rho \bmod \mathfrak{m}_A \text{ and } \det(\rho) = \chi \end{matrix} \right\} \simeq \{(x, y, a, s) \in \mathfrak{m}_A^4 : (1 + s)^{p^m} = 1\}$$
$$\rho_{(x,y,a,s)} \leftarrow (x, y, a, s),$$

where

$$\rho(\varphi) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + a & \\ & \chi(\varphi)/(\alpha + a) \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}$$

$$\rho(\sigma) = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 + s & \\ & (1+s)^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}. \qquad \square$$

3. Taylor's *Ihara avoidance* deformations

Now, a natural question is:

Question 3.0.1. What happens when $\overline{\rho}(\mathrm{Fr}_K)$ has an eigenvalue with multiplicity?

Of course, it suffices to treat the case when $\overline{\rho}$ is trivial (since one can twist by a central character). Thus, to recap, our assumptions now are:

- $K/\mathbb{Q}_\ell$ is a finite extension
- $\overline{\rho}\colon G_K \to \mathrm{GL}_2(k_L)$ is the trivial representation
- $q_L \equiv 1 \pmod p$
- χ is unramified, and $\overline{\chi} = 1$.

Now, again $\rho^\square$ is tamely ramified, so $\rho^\square$ is determined by $\rho^\square(\sigma)$ and $\rho^\square(\varphi)$, by Remark 2.0.2.

Now, let us make the following definition:

Definition 3.0.2. (1) Let $\mathcal{P}_{\mathrm{ur}}$ be the minimal ideal of $R^\square_{\overline{\rho},\chi}$ such that $\rho^\square(\sigma) = I_2 \pmod{\mathcal{P}_{\mathrm{ur}}}$. In other words, writing $\rho^\square(\sigma) = \begin{pmatrix} 1+x & y \\ z & 1+w \end{pmatrix}$ for $x, y, z, w \in \mathfrak{m}^\square_{\overline{\rho},\chi}$, we let $\mathcal{P}_{\mathrm{ur}} = (x, y, z, w) \subset \mathfrak{m}^\square_{\overline{\rho},\chi}$.

(2) For any root of unity $\zeta \in \mathcal{O}_K^\times$, let $\mathcal{P}_\zeta$ be the minimal ideal of $R^\square_{\overline{\rho},\chi}$ modulo which $\rho^\square(\sigma)$ has characteristic polynomial $(X-\zeta)(X-\zeta^{-1})$. In other words, $\mathcal{P}_\zeta = (\operatorname{tr}\rho^\square(\sigma) - \zeta - \zeta^{-1}, \det\rho^\square(\sigma) - 1)$.

(3) Let $\mathcal{P}_m$ be the minimal ideal of $R^\square_{\overline{\rho},\chi}$ modulo which $\rho^\square(\sigma)$ has characteristic polynomial $(X-1)^2$, and $q_K(\operatorname{tr}\rho^\square(\sigma))^2 = (1+q_K)^2 \det\rho^\square(\varphi)$.

Write $R^\square_{\overline{\rho},\chi,\bullet}$ for $R^\square_{\overline{\rho},\chi}/\mathcal{P}_\bullet$.

Remark 3.0.3. The relation in (3) holds in particular when $\rho^\square(\sigma) = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.

The ideals defined above have nice ring-theoretic properties:

Proposition 3.0.4. *The minimal primes of $R^\square_{\overline{\rho},\chi}$ are precisely $\sqrt{\mathcal{P}_{\mathrm{ur}}}$, $\sqrt{\mathcal{P}_m}$, and $\sqrt{\mathcal{P}_\zeta}$ for $\zeta \neq 1$. Moreover, $\sqrt{\mathcal{P}_1} = \sqrt{\mathcal{P}_{\mathrm{ur}}} \cap \sqrt{\mathcal{P}_m}$.*

Now, we have [Gee22, Theorem 3.38]:

Theorem 3.0.5. *We have $R^\square_{\overline{\rho},\chi,1}/\lambda = R^\square_{\overline{\rho},\chi,\zeta}/\lambda$. Furthermore,*

(1) If $\zeta \neq 1$ the $R^\square_{\overline{\rho},\chi,\zeta}[1/p]$ is geometrically irreducible of dimension 3

(2) $R^\square_{\overline{\rho},\chi,\mathrm{ur}}$ is formally smooth over $\mathcal{O}_L$ (and thus geometrically irreducible) of relative dimension 3

(3) $R^{\square}_{\overline{\rho},\chi,m}[1/p]$ *is geometrically irreducible of dimension* 3.

(4) $\mathrm{Spec} R^{\square}_{\overline{\rho},\chi,1} = \mathrm{Spec} R^{\square}_{\overline{\rho},\chi,\mathrm{ur}} \cup \mathrm{Spec} R^{\square}_{\overline{\rho},\chi,m}$ *and* $\mathrm{Spec} R^{\square}_{\overline{\rho},\chi,1}/\lambda = \mathrm{Spec} R^{\square}_{\overline{\rho},\chi,\mathrm{ur}}/\lambda \cup \mathrm{Spec} R^{\square}_{\overline{\rho},\chi,\mathrm{m}}/\lambda$ *are both a union of two irreducible components, and have relative dimension* 3.

References

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