MODULARITY SEMINAR: PROOF OF THE LANGLANDS–TUNNELL THEOREM

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Abstract. In these notes, we give the proof of the Langlands–Tunnell theorem.

These notes have been plagiarized to the highest order from Gelbart's article [Gel97].

1. Introduction

Today we will prove the following theorem:

Theorem 1.1. *Suppose F is a number field and the irreducible representation*

$$\sigma : W_F \to \mathrm{GL}_2(\mathbb{C})$$

has solvable image in $\mathrm{PGL}_2(\mathbb{C})$. *Then there exists a (unique) irreducible automorphic cuspidal representation* $\pi(\sigma) = \bigotimes_v \pi_v$ *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that* $\mathrm{trace}(\sigma(\mathrm{Frob}_v)) = \mathrm{trace}(t_{\pi_v})$ *for almost all v.*

Here, W_F is the Weil group of F and t_{π_v} is the *Satake parameter* of π_v, defined to be the element

$$t_{\pi_v} = \begin{pmatrix} \mu_1(\varpi_v) & 0 \\ 0 & \mu_2(\varpi_v) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

where μ_1, μ_2 are the unramified characters of $F_v^\times$ inducing π_v as a principal series.

Now, Theorem 1.1 lends easily to the following formulation of the Langlands–Tunnell theorem:

Theorem 1.2 (Langlands–Tunnell, automorphic version). *Suppose $\sigma : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{C})$ is a continuous, irreducible, two-dimensional representation whose image in $\mathrm{PGL}_2(\mathbb{C})$ is solvable. Suppose also that σ is odd in the sense that*

$$\det(\sigma(\tau)) = -1 \tag{1.1}$$

(where $\tau \in G_\mathbb{Q}$ is the automorphism defined by complex conjugation). Then there exists an irreducible automorphic cuspidal representation $\pi(\sigma) = \bigotimes_p \pi_p$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ which is of weight 1, central character $\det \sigma$, *and such that for almost all p, $\pi_p = \pi(\mu_1, \mu_2)$ is unramified with* $\mathrm{trace}(\sigma(\mathrm{Frob}_p)) = \mu_1(p) + \mu_2(p)$.

Proof. Applying Theorem 1.1 to the case where $F = \mathbb{Q}$ and σ factors through $G_\mathbb{Q}$, there is procured an irreducible representation $\pi = \pi(\sigma)$ with the correct local properties at almost all primes. It remains to verify that π_∞ has the correct weight 1 and central character. Namely, we want $\pi_\infty = \pi(1, \mathrm{sgn})$. Equivalently, we want $\sigma_\infty = 1 \oplus \mathrm{sgn}$. But when viewed as a representation of $W_\mathbb{R}$, σ_∞ is trivial on $\mathbb{C}^\times$. Thus σ_∞ cannot be induced from a nontrivial character of $\mathbb{C}^\times$. Thus σ_∞ cannot be a two-dimensional representation, so it is the sum of two characters μ_i, with $\mu_i \sim (t_i, \varepsilon_i)$. Since σ_∞ is trivial on $\mathbb{C}^\times$, it follows that $t_i = 0$.

Date: September 28, 2023.

On the other hand, the assumption $\det\sigma(\tau) = -1$ implies that $\sigma(\tau)$ is not a scalar, hence $\sigma(\tau) \sim \mathrm{diag}(-1,1)$, so $\pi_\infty = \mathrm{Ind}\,1 \cdot \mathrm{sgn}$, as desired. □

Moreover, recall that there is a way to convert between normalized new forms and irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. So these theorems can be phrased in terms of modular forms over $\mathbb{Q}$ whose Fourier coefficients satisfy certain congruence conditions at the primes.

Corollary 1.3 (Langlands–Tunnell, classical version)**.** *Suppose $\sigma : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{C})$ is a continuous, irreducible, two-dimensional representation whose image in $\mathrm{PGL}_2(\mathbb{C})$ is solvable. Suppose also that σ is odd in the sense of (1.1). Then there exists a normalized*

$$f(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z} \in \mathcal{S}_1(\Gamma_0(N), \psi)$$

(for some N and ψ), such that f is an eigenform for all the Hecke operators, and $b_q = \mathrm{trace}(\sigma(\mathrm{Frob}_q))$ for almost all primes q.

At the end of the notes, we will use the Langlands–Tunnell theorem to complete the proof that if $p = 3$, and if the mod p Galois representation $\overline{\rho}_{E,3}$ of the elliptic curve E over $\mathbb{Q}$ is irreducible, then it is modular.

2. About the proof of Theorem 1.1

Because any continuous representation $\sigma : G_F \to \mathrm{GL}_2(\mathbb{C})$ factors through some finite extension, it must be semisimple, its image in $\mathrm{GL}_2(\mathbb{C})$ is finite, and its image in $\mathrm{PGL}_2(\mathbb{C})$ is one of the symmetry groups of a regular polyhedron in $\mathbb{R}^3$. The image of any such σ is one of the following:

- A cyclic group C_n (iff σ is reducible);
- A dihedral group D_n of order $2n$ (iff σ is induced from the nontrivial quadratic character on some G_K, for K/F a quadratic extension);
- The tetrahedral group A_4;
- The octahedral group S_4;
- The isocahedral group A_5.

In Theorem 1.1, we only consider those representations which are both irreducible and solvable. So we are precisely interested in the dihedral, tetrahedral, and octahedral cases.

The proof will use several basic instances of Langlands functoriality for the global case. Here, we recall the ones we need.

2.1. Automorphic induction for quadratic and cyclic extensions.

Let K/F be a cyclic extension of number fields of degree n. For all applications, we will only need the cases $n = 2$ or 3.

Theorem 2.1. *For each Hecke character χ of K there is an automorphic representation $\pi(\chi)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ whose L-function $L^S(s,\pi)$ equals the Hecke L-function $L^S(s,\chi)$. Moreover, $L^S(s,\pi(\chi))$ is entire (and hence $\pi(\chi)$ is cuspidal automorphic) if χ does not factor through the norm map $\mathrm{N}_{K/F}$ (equivalently χ is not fixed by the action of $\mathrm{Gal}(K/F)$).*

For $n = 2$ and $F = \mathbb{Q}$, this follows from classical work of Hecke and Maass; for $n = 2$ or 3 and F arbitrary, this is proved in [JL70] and [JPSS79]

2.2. **Symmetric square lifting from** $\mathrm{GL}(2)$ **to** $\mathrm{GL}(3)$**.** Let A be the three-dimensional representation of $\mathrm{PGL}_2(\mathbb{C})$ determined by the adjoint action of $\mathrm{PGL}_2(\mathbb{C})$ on the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$, and denote the resulting three-dimensional representation

$$\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) \xrightarrow{A} \mathrm{GL}_3(\mathbb{C})$$

of $\mathrm{GL}_2(\mathbb{C})$ by Ad.

Theorem 2.2. *We have the following.*

(a) *To each cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$*, there exists an automorphic representation* Π *of* $\mathrm{GL}_3(\mathbb{A}_F)$ *such that for almost all* v*,*

$$\Pi_v = \Pi_v(\mathrm{Ad}(\sigma_v)),$$

whenever $\pi_v = \pi_v \circ \sigma_v$*; equivalently,* $t_{\Pi_v} = \mathrm{Ad}(t_{\pi_v})$*.*

(b) *The lift of* π *to* $\mathrm{GL}(3)$ *is* cuspidal *automorphic unless* π *is monomial, i.e. of the form* $\pi(\sigma)$*, with* σ *induced from a Hecke character of some quadratic extension* K*.*

See [GJ78].

2.3. **Rankin–Selberg convolution for** $\mathrm{GL}(3) \times \mathrm{GL}(3)$**.** We have the following.

Theorem 2.3. *Given cuspidal representations* π *and* π' *on* $\mathrm{GL}(3)$*, let* $L^S(s, \pi \times \pi')$ *denote the partial* L*-function*

$$\prod_{v \notin S} \det(I - (t_{\pi_v} \otimes t_{\pi'_v}) q^{-s})^{-1}.$$

(a) $L^S(s, \pi \times \pi')$ *extends to a meromorphic function in* $\mathbb{C}$*, satisfying a functional equation as* $s \mapsto 1 - s$*.*

(b) $L^S(s, \pi \times \pi')$ *may be completed to an Euler product*

$$L(s, \pi_v \times \pi'_v) = \prod_v L(s, \pi_v \times \pi'_v)$$

which is holomorphic on $\mathrm{Re}(s) \geq 1$ *except for a pole at* s *with* $\mathrm{Re}(s) = 1$ *if and only if* $|\det()|^{s-1} \otimes \pi \simeq \widetilde{\pi}'$*, the contragredient of* π'*.*

See [JPSS81], [JPSS79], and [MgW89]. This gives us a way of relating π and $\widetilde{\pi}$ from knowing the behavior of the L-functions.

2.4. **Base change for** $\mathrm{GL}(2)$**.** Arguably the most important incarnation of Langlands functoriality to the Langlands–Tunnell setting is the theory of base change lifting for $\mathrm{GL}(2)$-representations. We discussed this theory in last week's lecture, and we restate the main ideas here. First, fix E a cyclic extension of the number field F, of prime degree.

Proposition–Definition 2.4. *Suppose* $\pi = \otimes_v \pi_v$ *is an automorphic cuspidal representation of* $\mathrm{GL}_2(\mathbb{A}_F)$*, and* $\Pi = \otimes_w \Pi_w$ *is an automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_E)$*. Then* Π *is a* base change lift *of* π*, denoted* $\mathrm{BC}_{E/F}(\pi)$*, if for each place* v *of* F*, and* $w \mid v$*, the Langlands parameter attached to* Π_w *equals the restriction to* W_{E_w} *of the Langlands parameter* $\sigma_v : W_{F_v} \to \mathrm{GL}_2(\mathbb{C})$ *of* π_v*.*

(a) *Every cuspidal representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *has a unique base change lift to* $\mathrm{GL}_2(\mathbb{A}_E)$*; the lift is itself cuspidal, except in the case where* E/F *is quadratic and* π *is monomial of the form* $\pi(\sigma)$*, with* $\sigma = \mathrm{Ind}_{W_E}^{W_F} \theta_{E/F}$*.*

(b) *If two cuspidal representations* π *and* π' *have the same base change lift to* E*, then* $\pi' \simeq \pi \otimes \omega$ *for some character* ω *of* $F^\times \mathrm{N}_{E/F}(\mathbb{A}_E^\times) \backslash \mathbb{A}_F^\times$*.*

(c) *A cuspidal representation* Π *of* $\mathrm{GL}_2(\mathbb{A}_E)$ *equals* $\mathrm{BC}_{E/F}(\pi)$ *for some cuspidal* π *on* $\mathrm{GL}_2(\mathbb{A}_F)$ *if and only if* Π *is invariant under the natural action of* $\mathrm{Gal}(E/F)$*.*

See [Lan80]. In the course of the proof, we will also need a version of base change for cubic, but not necessarily Galois extensions. See below.

3. The dihedral case

In the case where $\sigma : W_F \to \mathrm{GL}_2(\mathbb{C})$ has dihedral image in $\mathrm{PGL}_2(\mathbb{C})$, it must be a *monomial representation*; that is, a representation $\mathrm{Ind}_{W_E}^{W_F}\theta_{E/F}$ induced from the nontrivial quadratic Hecke character $\theta_{E/F}$ over E associated to some quadratic extension E/F. (It is pretty much a group-theoretic fact that these conditions are equivalent, but I neither have a proof nor will try).

Thus, this case reduces precisely to the statement of automorphic induction, Theorem 2.1.

4. The tetrahedral case

We are given an irreducible representation $\sigma : W_F \to \mathrm{GL}_2(\mathbb{C})$, and wish to construct a cuspidal representation $\pi(\sigma)$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\mathrm{trace}(\sigma(\mathrm{Frob}_v)) = \mathrm{trace}(t_{\pi_v})$ for almost every v. Suppose that there is such a representation. For a cyclic extension E/F of number fields, the base change lifting is defined by

$$\mathrm{BC}_{E/F}(\pi(\sigma)) = \pi(\mathrm{Res}_{W_E}^{W_F}\sigma).$$

Thus, to look for candidates for $\pi(\sigma)$, we look among cuspidal π for which $\mathrm{BC}_{E/F}(\pi) = \pi(\mathrm{Res}_{W_E}^{W_F}\sigma)$.

First, we need our extension E. Consider the solvable group A_4 exhibiting the composition series $A_4 \rhd D_2 \rhd 1$. Since $A_4/D_2 \simeq \mathbb{Z}/3\mathbb{Z}$, the inverse image of D_2 in W_F under the map $W_F \to A_4 \subset \mathrm{PGL}_2(\mathbb{C})$ is a normal subgroup of index 3, hence the Weil group of a cubic extension E of F. We have the following diagram:

$$\begin{CD}
1 @>>> W_E @>>> W_F @>>> \mathrm{Gal}(E/F) @>>> 1\\
@. @VVV @VVV @VVV @.\\
1 @>>> D_2 @>>> A_4 @>>> \mathbb{Z}/3\mathbb{Z} @>>> 1
\end{CD}$$

Thus, the resulting representation $\sigma_E : W_E \to \mathrm{GL}_2(\mathbb{C})$ is dihedral, i.e. monomial. Let $\pi(\sigma_E)$ be the automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ attached to this monomial representation. This is invariant under the action of $\mathrm{Gal}(E/F)$; indeed $\pi(\sigma_E)^\tau = \pi(\sigma_E^\tau) = \pi(\sigma_E)$. So by Proposition–Definition 2.4, $\pi(\sigma_E)$ will be $\mathrm{BC}_{E/F}(\pi_i)$ for exactly three classes of cuspidal representations π_i of $\mathrm{GL}_2(\mathbb{A}_F)$, each related to each other by a twist $\omega \circ \det$ where ω is a character of $F^\times \mathrm{N}_{E/F}(\mathbb{A}_E^\times)\backslash\mathbb{A}_F^\times \simeq \mathrm{Gal}(E/F)$, i.e. $\pi_j \simeq \pi_i \otimes \omega \circ \det$.

Recall that the central character of $\pi(\sigma)$ is to be $\det\sigma$. On the other hand, the central character ω_i of each π_i above "base change lifts" to the central character of $\pi(\sigma_E)$, which is $\det\sigma_E = (\det\sigma) \circ \mathrm{N}_{E/F}$. Since each $\omega_i = \omega_j\omega^2$ if $\pi_i = \pi_j \otimes \omega \circ \det$, exactly one of these π_i has central character $\det\sigma$; we call this $\pi_{\mathrm{ps}}(\sigma)$. We claim that this is the desired $\pi(\sigma)$.

Write $\pi_{\mathrm{ps}}(\sigma) = \otimes_v \pi_v$. Then for each v, $\pi_v = \pi_v(\sigma'_v)$ for some $\sigma'_v : W_{F_v} \to \mathrm{GL}_2(\mathbb{C})$. We wish to prove that $\sigma'_v = \sigma_v$ for almost every v. By construction,

$$(\sigma_E)_w = \mathrm{Res}_{W_{E_w}}^{W_{F_v}}\sigma'_v = \mathrm{Res}_{W_{E_w}}^{W_{F_v}}\sigma_v. \tag{4.1}$$

There is nothing to prove when v splits completely in E, so we assume that E_w/F_v is cubic and unramified.

If Frob_v is a Frobenius element of $\mathrm{Gal}(E_w/F_v)$, we suppose (up to conjugacy) that

$$\sigma_v(\mathrm{Frob}_v) = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} \text{ and } \sigma_v'(\mathrm{Frob}_v) = \begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix}$$

for some $a_v, b_b, c_v, d_v \in \mathbb{C}^\times$. Since σ_v is completely determined by where it maps the Frobenius conjugacy class, it will suffice to prove that these are conjugate. But since σ_v and σ_v' have the same restriction to W_{E_w}, and $\mathrm{Frob}_v^3 \in W_{E_w}$, we must have $\sigma_v(\mathrm{Frob}_v)^3$ conjugate to $\sigma_v'(\mathrm{Frob}_v)^3$. In particular, for some pair of cube roots of 1, say ξ and ξ', either

$$c_v = \xi a_v \text{ and } d_v = \xi' b_v$$

or else.

$$c_v = \xi b_v \text{ and } d_v = \xi' a_v.$$

We claim now that $\xi' = \xi^2$. Indeed, $\pi_{\mathrm{ps}}(\sigma)$ was chosen so that $\omega_{\pi_{\mathrm{ps}}}(\sigma) = \det \sigma$. Since this implies $\det \sigma_v' = \det \sigma_v$, we must have $\xi\xi' = 1$, i.e. $\xi' = \xi^2$.

To complete the proof, it will suffice to prove that $\xi = 1$. For now, let us assume the following:

Claim 4.1. $\mathrm{Ad} \circ \sigma_v' = \mathrm{Ad} \circ \sigma_v$.

Since the kernel of $\mathrm{Ad} : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$ is precisely the group of scalar matrices, it follows that $\sigma_v(\mathrm{Frob}_v)$ and $\sigma_v'(\mathrm{Frob}_v)$ must differ by some scalar $\lambda \neq 0$. Thus

$$\begin{pmatrix} \xi a_v & 0 \\ 0 & \xi^2 b_v \end{pmatrix} \text{ is conjugate to } \begin{pmatrix} \lambda a_v & 0 \\ 0 & \lambda b_v \end{pmatrix}.$$

If $(\lambda a_v, \lambda b_v) = (\xi a_v, \xi^2 b_v)$, then $\lambda = \xi = \xi^2 = 1$ because ξ is a cube root of unity. On the other hand, if $(\lambda a_v, \lambda b_v) = (\xi^2 b_v, \xi a_v)$, then $\lambda^2 = 1$. If $\lambda = -1$, then the image of

$$\sigma_v(\mathrm{Frob}_v) = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} = \begin{pmatrix} a_v & 0 \\ 0 & a_v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \xi\lambda \end{pmatrix}$$

in $\mathrm{PGL}_2(\mathbb{C})$ is of order 6. But A_4 has no elements of order 6, contradiction. Thus $\lambda = 1$, and we are done!

4.1. **Proof of Claim 4.1.** Observe that $\mathrm{Ad} \circ \sigma : W_F \to \mathrm{GL}_2(\mathbb{C})$ is a monomial representation; that is, there is a character θ of W_E *not* invariant under $\mathrm{Gal}(E/F)$ such that

$$\mathrm{Ad} \circ \sigma = \mathrm{Ind}_{W_E}^{W_F} \theta.$$

By Theorem 2.1 in the case $n = 3$, we have $\mathrm{Ad}\circ$ associated to a *cuspidal* automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, call it Π_1. On the other hand, $\pi_{\mathrm{ps}}(\sigma)$ has a symmetric square lift to $\mathrm{GL}_2(\mathbb{A}_F)$, call it Π_1^*, which is almost everywhere associated to the Langlands parameter $\mathrm{Ad} \circ \sigma_v'$. Moreover, this Π_1^* is also cuspidal.

Lemma 4.2. *The Rankin–Selberg L-function $L(s, \Pi_1^* \times \widetilde{\Pi}_1)$ on $\mathrm{GL}(3) \times \mathrm{GL}(3)$ has a pole at $s = 1$.*

By Theorem 2.3, we are completely done if we prove this.

Proof. We claim that for almost all places v,

$$L(s,(\Pi_1^*)_v\times(\widetilde{\Pi}_1)_v)=L(s,(\Pi_1)_v\times(\widetilde{\Pi}_1)_v).$$

This suffices, because then $L(s,\Pi_1^*\times\widetilde{\Pi}_1)=L(s,\Pi_1\times\widetilde{\Pi}_1)$ after ignoring nonzero terms. However, the RHS has a pole at $s=1$ by Theorem 2.3, so the LHS does too, and thus $\Pi_1^*\simeq\Pi_1$.

For almost every v we have

$$L(s,(\Pi^*)_v\times(\widetilde{\Pi})_v)=L(s,(\mathrm{Ad}\circ\sigma_v')\otimes(\mathrm{Ad}\circ\sigma_v)).$$

Since $\mathrm{Ad}\circ\sigma$ is induced from θ on E, we have

$$\mathrm{Ad}(\widetilde{\sigma}_v)=\bigoplus_{w|v}\mathrm{Ind}_{W_{E_w}}^{W_{F_v}}\theta_w^{-1}.$$

Hence, if $\Sigma_w=\mathrm{Res}_{W_{E_w}}^{W_{F_v}}\mathrm{Ad}(\sigma_v)$, $\Sigma_w'=\mathrm{Res}_{W_{E_w}}^{W_{F_v}}\mathrm{Ad}(\sigma_v')$, we get

$$\mathrm{Ad}(\sigma_v)\otimes\mathrm{Ad}(\widetilde{\sigma}_v)=\bigoplus_{w|v}\mathrm{Ind}_{W_{E_w}}^{W_{F_v}}(\theta_w^{-1}\otimes\Sigma_w)$$

and

$$\mathrm{Ad}(\sigma_v')\otimes\mathrm{Ad}(\widetilde{\sigma}_v)=\bigoplus_{w|v}\mathrm{Ind}_{W_{E_w}}^{W_{F_v}}(\theta_w^{-1}\otimes\Sigma_w')$$

(This comes from the reciprocity relation $\sigma\otimes\mathrm{Ind}_H^G\Sigma\simeq\mathrm{Ind}_H^G(\mathrm{Res}_H^G\sigma\otimes\Sigma)$.) So since $\Sigma_w\simeq\Sigma_w'$ (see (4.1)), we indeed have

$$\begin{aligned}L(s,(\Pi^*)_v\times(\widetilde{\Pi})_v)&=L(s,(\mathrm{Ad}\circ\sigma_v')\otimes(\mathrm{Ad}\circ\sigma_v))\\&=L(s,(\mathrm{Ad}\circ\sigma_v)\otimes(\mathrm{Ad}\circ\sigma_v))\\&=L(s,(\Pi)_v\times(\widetilde{\Pi})_v),\end{aligned}$$

and we are done. □

The proof of this case is due to Langlands [Lan80].

5. The octahedral case

Consider the group S_4 which contains A_4 as a normal subgroup of index 2. As before, consider the diagram The resulting representation $\sigma_E: W_F\to\mathrm{GL}_2(\mathbb{C})$ is tetrahedral. Since

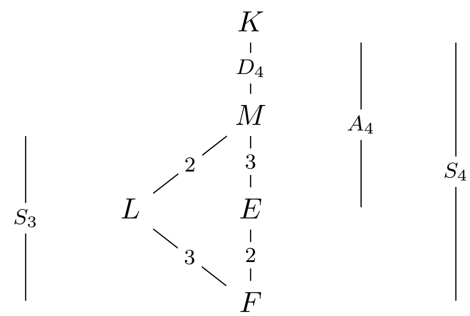
$$\begin{array}{ccccccccc}1&\longrightarrow&W_E&\longrightarrow&W_F&\longrightarrow&\mathrm{Gal}(E/F)&\longrightarrow&1\\&&\downarrow&&\downarrow&&\downarrow&&\\1&\longrightarrow&A_4&\longrightarrow&S_4&\longrightarrow&\mathbb{Z}/2\mathbb{Z}&\longrightarrow&1\end{array}$$

we have just proved 1.1 for that case, let $\pi(\sigma_E)$ be the automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ attached to this monomial representation. This is invariant under $\mathrm{Gal}(E/F)$. So $\pi(\sigma_E)$ will be $\mathrm{BC}_{E/F}(\pi_i)$ for exactly two classes of cuspidal representations π_i, each related to each other by a twist $\omega\circ\det$ where ω is a character of $F^\times\mathrm{N}_{E/F}(\mathbb{A}_E^\times)\backslash\mathbb{A}_F^\times\simeq\mathrm{Gal}(E/F)$. We will have $\omega_i=\omega_j\omega^2$, but the issue here is that now $\omega^2=1$, and thus ω_i cannot be determined by its central character!

Instead, we appeal to a version of base change for cubic, non-Galois extensions:

Proposition 5.1. *If L is a cubic not necessarily Galois extension of F, then each automorphic cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ has a base change lift Π on $\mathrm{GL}_2(\mathbb{A}_L)$, i.e. $\Pi = \mathrm{BC}_{L/F}(\pi)$ is automorphic, and for almost every place v of F, and a place w of L dividing v, $\pi_v = \pi_v(\sigma_v)$ implies $\Pi_w = \pi_w(\mathrm{Res}_{L_w/F_v}\,\sigma_v)$.*

Now, introduce L/F as the cubic, non-Galois subextension of K/F fixed by a 2-Sylow subgroup (of order 8) of S_4. More precisely, L is the fixed field of all elements of $\mathrm{Gal}(K/F)$ that map to this chosen subgroup. Then if M is the composition in K of L and E (the quadratic Galois extension chosen above), we have the tower of fields:



Notice that σ_L is monomial, so $\pi(\sigma_L)$ exists.

Lemma 5.2. *There exists a unique index $i = 1, 2$ such that $\mathrm{BC}_{L/F}(\pi_i) = \pi(\sigma_L)$.*

Proof. By Proposition 5.1, $\mathrm{BC}_{L/F}(\pi_i)$ exists for $i = 1, 2$. By transitivity of base change,

$$\mathrm{BC}_{M/L}(\mathrm{BC}_{L/F}(\pi_i)) = \pi(\sigma_M).$$

Since $\mathrm{BC}_{L/F}(\pi_i)$ have the same quadratic base change to M, it follows from Proposition–Definition 2.4 that

$$\mathrm{BC}_{L/F}(\pi_2) \simeq \mathrm{BC}_{L/F}(\pi_1) \otimes \omega_{M/L}.$$

The representations $\mathrm{BC}_{L/F}(\pi_i)$ are distinct for $i = 1, 2$, for if

$$\mathrm{BC}_{L/F}(\pi_1) \simeq \mathrm{BC}_{L/F}(\pi_1) \otimes \omega_{M/L},$$

then by [Lan80, Lemma 11.7], π_1 is a monomial representation. This would imply $\mathrm{BC}_{M/L}(\mathrm{BC}_{L/F}(\pi_i)) = \pi(\sigma_M)$ is not cuspidal. But the image of σ_M in $\mathrm{PGL}_2(\mathbb{C})$ is $S_3 \simeq D_3$, which means that σ_M is irreducible, i.e. $\pi(\sigma_M)$ is cuspidal. Thus, $\mathrm{BC}_{L/F}(\pi_1)$ and $\mathrm{BC}_{L/F}(\pi_2)$ are the two cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_L)$ that yield $\pi(\sigma_M)$ upon base change to M. Since we also have $\mathrm{BC}_{M/L}(\pi(\sigma_L)) = \pi(\sigma_M)$, it must be that $\pi(\sigma_L) = \mathrm{BC}_{L/F}(\pi_i)$ for exactly one i, as desired. □

The aforementioned π_i will be our candidate $\pi_{\mathrm{ps}}(\sigma)$. Write $\pi_{\mathrm{ps}}(\sigma) = \otimes_v \pi(\sigma'_v)$ as before. One proves exactly as in the tetrahedral case (but without having to take a lift to $\mathrm{GL}_3(\mathbb{C})$) that the non-existence of an element of order 6 in S_4 implies $\sigma_v \simeq \sigma'_v$ for almost all v.

The octahedral case is due to Tunnell [Tun81].

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