STAGE Shimura Notes

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These are notes on "Complex Multiplication, Shimura-Taniyama formula", written for my Spring 2023 STAGE talk. They reflect my understanding (or lack thereof) of the material, so are far from perfect. They are likely to contain some typos and/or mistakes, but ideally none serious enough to distract from the mathematics. With that said, enjoy and happy mathing.

In reality I finished and posted these notes months after I gave my talk, so this statement is ahistoric.

I should also mention that these notes were largely written for my own benefit, as a place to help me organize my understanding of some of the basics of Shimura varieties. As such, while the main body of the text is focused on the material directly relevant to my talk, there are also extra bits included in various appendices.

Contents

Introduction

The goal of this talk is to build up the statement of the main theorem of complex multiplication for abelian varieties, following [Mil17, Chapters 10 and 11]. Given an abelian variety A with CM by a field E, this theorem will describe the "Galois action on A and its torsion points." We include quotation marks here because the Galois group acting on A will not be attached to E, but instead to the so-called "reflex field" E^* of A, described later.

Why do we care about this Galois action in this seminar? Our last goal for the semester will be to define and construct canonical models of Shimura varieties. That is, given a Shimura datum (G, X), we have defined certain complex algebraic varieties

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

for $K \subset G(\mathbb{A}_f)$ a (sufficiently small) compact open. In the next talk, we wish to show that these $\mathrm{Sh}_K(G, X)(\mathbb{C})$'s are canonically the $\mathbb{C}$-points of some algebraic varieties $\mathrm{Sh}_K(G, X)$ define over some *number field* $E(G, X)$.

In certain cases, $\mathrm{Sh}_K(G, X)(\mathbb{C})$ is, in a natural way, a moduli variety over $\mathbb{C}$. In such cases, one can descend it to an algebraic variety over a number field by descending the corresponding moduli problem.

Example 1. If (G, X) is the Siegel Shimura datum, then $\mathrm{Sh}_K(G, X)(\mathbb{C})$ parameterizes (isomorphism classes of) abelian varieties/$\mathbb{C}$ equipped with level structure described by K. In this case, Sh_K will have a canonical model, over $\mathbb{Q}$, which is still a moduli space for abelian varieties equipped with appropriate level structure. △

However, not all Shimura varieties are known to solve moduli problems which easily descend to number fields. Thus, a difference strategy is required if one want to prove the existence of canonical models for general Shimura varieties. This is where complex multiplication comes in. Given a variety $V/\mathbb{C}$. To descend it to some subfield $E \subset \mathbb{C}$, one could extend the functor it represents from $\mathbb{C}$-schemes to E-schemes (as in the case of descending Siegel modular varieties), or one could define an action of $\mathrm{Aut}(\mathbb{C}/E)$ on V which is compatible with its action on $\mathbb{C}$ (see [Mil17, Corollary 14.6] for a precise statement). It is the latter approach which works for general Shimura varieties. In this context, the 'canonicity' of canonical models comes from specifying their action on a (dense) subset of so-called "special" points (analogous to the CM points on Siegel modular varieties).

1 Abelian Varieties of CM type

Our first task is to define our objects of study.

Definition 1.2. A `CM field` is a number field E which totally imaginary, quadratic extension of a totally real field F. Each embedding $F \hookrightarrow \mathbb{R}$ extends into a conjugate pair of embeddings $E \hookrightarrow \mathbb{C}$. A `CM-type` Φ for E is a choice such an extension for each $F \hookrightarrow \mathbb{R}$; that is, it is a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that

$$\mathrm{Hom}(E, \mathbb{C}) = \Phi \sqcup \overline{\Phi}$$

($\overline{\Phi} = \{\overline{\varphi} : \varphi \in \Phi\}$). ⋄

Notation 1.3. Given a CM field E with maximal totally real subfield F, we set

$$E_{\mathbb{C}} = E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \prod_{\varphi: E \hookrightarrow \mathbb{C}} \mathbb{C}_{\varphi}, \quad \mathbb{C}_{\varphi} = \mathbb{C}$$

$$E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma : F \hookrightarrow \mathbb{R}} \mathbb{C}_\sigma, \quad \mathbb{C}_\sigma = E \otimes_{F,\sigma} \mathbb{R}$$
$$F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma : F \hookrightarrow \mathbb{R}} \mathbb{R}_\sigma, \quad \mathbb{R}_\sigma = \mathbb{R}.$$

Example 1.4. If $E = \mathbb{Q}(\sqrt{-d})$ is quadratic imaginary, then E is CM with maximal totally real subfield $F = \mathbb{Q}$. In this case, a CM-type for E is simply a choice of embedding $E \hookrightarrow \mathbb{C}$. △

Notation 1.5. Given a CM-type Φ, we let $\mathbb{C}^\Phi$ denote that $(E \otimes_{\mathbb{Q}} \mathbb{C})$-module

Definition 1.6. An ``‘‘isogeny’’`` of abelian varieties $\alpha : A \to B$ is an invertible element $\alpha \in \operatorname{Hom}^0(A, B) = \operatorname{Hom}(A, B) \otimes \mathbb{Q}$, i.e. an isomorphism in the isogeny category. ⋄

Definition 1.7. Let E be a CM field of degree $2g$ over $\mathbb{Q}$. Let $A/\mathbb{C}$ be an abelian variety of dimension g, and let i be a homomorphism $E \to \operatorname{End}^0(A) := \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If

$$T_0(A) \simeq \mathbb{C}^\Phi \text{ as } E_{\mathbb{C}}\text{-modules}$$

for some CM-type Φ, then we say that (A, i) is `of CM-type` (E, Φ). ⋄

Remark 1.8. Let $A/\mathbb{C}$ be a g-dimensional abelian variety equipped with a morphism $E \to \operatorname{End}^0(A)$ for some degree $2g$ CM field E. Then, (A, i) will be of CM-type (E, Φ) for some Φ. Indeed, $A \cong T_0(A)/\operatorname{H}_1(A, \mathbb{Z})$ (via the Lie exponential $\exp : \operatorname{Lie} A \twoheadrightarrow A$) and

$$\operatorname{H}_1(A; \mathbb{C}) \simeq (\operatorname{H}_1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = T_0(A) \otimes_{\mathbb{R}} \mathbb{C} = T_0(A) \oplus \overline{T_0(A)}.$$

Above, $\operatorname{H}_1(A, \mathbb{Q})$ is a 1-dimensional E-vector space (compare dimensions over $\mathbb{Q}$), so $T_0(A) \oplus \overline{T_0(A)} \simeq \operatorname{H}_1(A, \mathbb{C}) \simeq E_{\mathbb{C}}$. Thus, the set of $\varphi : E \hookrightarrow \mathbb{C}$ occurring in $T_0(A)$ must form a CM-type. ◦

Example 1.9. The elliptic curve $A : y^2 = x^3 - x$ has CM by $E = \mathbb{Q}(i)$ where $i \in E$ acts via $[i] : (x, y) \mapsto (-x, iy)$. △

Example 1.10. Fix an odd prime p and let $E = \mathbb{Q}(\zeta_p)$. This field is CM, with totally real subfield $F = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Consider the hyperelliptic curve $C : y^2 = x^p + 1$, and let $A = \operatorname{Jac}(C)$. Then, ζ_p acts on C via $[\zeta_p] : (x, y) \mapsto (e^{2\pi i/p}x, y)$, and so one gets an induced map $E \to \operatorname{End}^0(A)$. Explicitly, this maps sends ζ_p to $[\zeta_p]^* : A \to A$. One can compute[1] that the associated CM-type here is

$$\Phi = \left\{ \zeta_p \mapsto e^{2\pi i k/p} : k = 1, \ldots, g \right\}. \qquad \triangle$$

Example 1.11. Let $A_1/\mathbb{C}$ be an elliptic curve with CM by an imaginary quadratic K. Let $A := A_1^g$, a g-dimensional abelian variety. Note that, because $K \hookrightarrow \operatorname{End}^0(A_1)$ by assumption, we have an embedding $M_g(K) = \operatorname{End}_K(K^g) \hookrightarrow \operatorname{End}^0(A)$. Now, let F be a totally real number field of degree g. After choosing a $\mathbb{Q}$-basis for F, we can embed $F \hookrightarrow M_g(\mathbb{Q}) \subset M_g(K)$ via its multiplication

[1] Look at the action of $[\zeta_p]$ on $\operatorname{H}^1(C, \Omega^1) = \operatorname{H}^1(A, \Omega^1) = T_0(A)^\vee$. A basis is given by $x^k \mathrm{d}x/y$ for $k = 0, \ldots, g-1$, and $[\zeta_p]^*(x^k \mathrm{d}x/y) = (e^{2\pi i/p}x)^k \mathrm{d}(e^{2\pi i/p}x)/y = e^{2\pi i(k+1)/p} x^k \mathrm{d}x/y$.

action on itself. Thus, $\mathrm{End}^0(A) \supset M_g(K)$ contains copies of both K and F, and so contains a copy of their compositum $E := KF$, a CM field of degree $2g$, so A has CM by E. △

Recall 1.12. Let $M = \mathbb{C}^n/\Lambda$ be a complex torus, and let J denote the induced complex structure on $V := \mathbb{R} \otimes_{\mathbb{Z}} \Lambda \simeq \mathbb{C}^n$. A `Riemann form` for M is an alternating form $E : \Lambda \times \Lambda \to \mathbb{Z}$ whose extension $E_{\mathbb{R}}$ to V satisfies

$$E(Ju, Jv) = E(u, v) \text{ for all } u, v \in V \text{ and } E_J(u, v) := E(u, Jv) \text{ is positive-definite}$$

(this is exactly a polarization of the integral Hodge structure $\mathrm{H}_1(M, \mathbb{Z})$). With this definition in place, M is an abelian variety if and only if it admits a Riemann form. ⊙

Proposition 1.13. *Let Φ be a CM-type on E. Abuse notation by letting Φ denote also the natural map $E \to \mathbb{C}^\Phi, a \mapsto (\varphi(a))_{\varphi \in \Phi}$. Then,*

(a) *The image $\Phi(\mathscr{O}_E) \subset \mathbb{C}^\Phi$ is a lattice.*

(b) *The quotient*

$$A_\Phi : \mathbb{C}^\Phi/\Phi(\mathscr{O}_E)$$

is an abelian variety of CM-type (E, Φ) for the natural homomorphism $i_\Phi : E \to \mathrm{End}^0(A_\Phi)$.

(c) *Any other pair (A, i) of CM-type (E, Φ) is E-isogenous to (A_Φ, i_Φ).*

Proof. **(a)** This holds simply because

$$\mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{R} \simeq E \otimes_{\mathbb{Q}} \mathbb{R} = E_{\mathbb{R}} \xrightarrow[\Phi]{\sim} \mathbb{C}^\Phi.$$

(b) $A_\Phi = \mathbb{C}^\Phi/\Phi(\mathscr{O}_E)$ is a complex torus by definition. To show that it is an abelian variety, one needs to show it supports a Riemann form. Choose some $\alpha \in E$ so that $\operatorname{Im}\varphi(\alpha) > 0$ for all $\varphi \in \Phi$.[2] We may scale α to assume that $\alpha \in \mathscr{O}_E$. Let $F \subset E$ be the maximal totally real subfield. Let $v \mapsto \overline{v}$ denote the nontrivial element of $\mathrm{Gal}(E/F)$. Then,

$$\psi(u, v) := \mathrm{Tr}_{E/\mathbb{Q}}(\alpha u \overline{v}) = \sum_{\varphi \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\Big(\varphi(\alpha)\varphi(u)\overline{\varphi(v)}\Big) \text{ for } u, v \in \mathscr{O}_E$$

is a Riemann form, so A_Φ is an abelian variety. Furthermore, $i_\Phi : \mathscr{O}_E \to \mathrm{End}(A_\Phi)$ sending $\alpha \in \mathscr{O}_E$ to multiplication by $\Phi(\alpha)$ ends to the desired $i_\Phi : E \to \mathrm{End}^0(A_\Phi)$. Finally, $T_0(A_\Phi) = \mathbb{C}^\Phi$ as $E_{\mathbb{C}}$-modules, by construction, so (A_Φ, i_Φ) is of CM-type (E, Φ).

(c) Say (A, i) is of CM-type (E, Φ). Then, $T_0(A) \simeq \mathbb{C}^\Phi$ as $E_{\mathbb{C}}$-modules, so $A \simeq \mathbb{C}^\Phi/\Lambda$ w/ $\mathbb{Q}\Lambda \subset \mathbb{C}^\Phi$ stable under the E-action. We must therefore have $\mathbb{Q}\Lambda = \Phi(E) \cdot \lambda$ for some $\lambda \in E_{\mathbb{R}}^\times$, so $\Lambda = \Phi(\mathscr{O}) \cdot \lambda$ for some lattice $\mathscr{O} \subset E$. Choose N such that $N\mathscr{O} \subset \mathscr{O}_E$. Then, we have isogenies

$$A = \mathbb{C}^\Phi/\Lambda = \mathbb{C}^\Phi/(\Phi(\mathscr{O}) \cdot \lambda) \xrightarrow{N} \mathbb{C}^\Phi/(\Phi(N\mathscr{O}) \cdot \lambda) \xleftarrow{\cdot\lambda} \mathbb{C}^\Phi/\Phi(\mathscr{O}_E).$$ ■

[2] Why possible?

We end with section with two nice properties of CM abelian varieties: they are always defined over number fields, and they have everywhere potentially good reduction.

Proposition 1.14. *Let (A, i) be an abelian variety of CM-type (E, Φ) over $\mathbb{C}$. Then, (A, i) has a model over $\overline{\mathbb{Q}}$, which is unique up to isomorphism.*

Proof. Uniqueness is easy: the functor $A \to_{\mathbb{C}}\colon \mathrm{AV}(\overline{\mathbb{Q}}) \to \mathrm{AV}(\mathbb{C})$ is fully faithful. Indeed, a morphism $A \to B$ between $\mathbb{C}$-abelian varieties is determined by its action on the (Zariski dense) subset of torsion points, but $A(\overline{\mathbb{Q}})_{\mathrm{tors}} = A(\mathbb{C})_{\mathrm{tors}}$, so any morphism is fixed by the action of $\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$, and so defined over $\overline{\mathbb{Q}}$.

For existence, suppose (A, i) is of CM-type (E, Φ). It clearly has a model over some subring $R \subset \mathbb{C}$ which is finitely generated over $\overline{\mathbb{Q}}$. Let $\mathfrak{m} \subset R$ be a maximal ideal where A has good reduction. Then, $R/\mathfrak{m} \simeq \overline{\mathbb{Q}}$, so (A, i) specializes to some (A', i') over $\overline{\mathbb{Q}}$, which is still of CM-type (E, Φ).[3] Proposition 1.13**(c)** shows that there is an isogeny $(A', i')_{\mathbb{C}} \to (A, i)$. Its kernel will be defined over $\overline{\mathbb{Q}}$ since $A'(\overline{\mathbb{Q}})_{\mathrm{tors}} = A'(\mathbb{C})_{\mathrm{tors}}$, so $(A'/H, i')$ is a model of (A, i) over $\overline{\mathbb{Q}}$. ∎

To show that CM abelian varieties have potentially everywhere good reduction, we use the Néron-Ogg-Shafarevich criterion [ST68, Theorem 1]. This says that A has good reduction if and only if its Tate module $T_\ell(A)$ is unramified for some (equivalently, any) prime away from the residue characteristic. We will find it more immediately useful to appeal to the following corollary.

Lemma 1.15 (Corollary 1, [ST68])**.** *Let F be a local field with residue characteristic p, and let $G_F := \mathrm{Gal}(F^s/F)$. Let A/F be an abelian variety. If, for some $\ell \neq p$, the image of G_F in $\mathrm{Aut}(T_\ell(A))$ is abelian, then A has potential good reduction at v.*

Proof. By Néron-Ogg-Shafarevich is suffices to show that inertia has finite image in $\mathrm{Aut}(T_\ell(A))$. Because the image is abelian, class field theory tells us that the image of inertia is a quotient of the group $\mathscr{O}_F^\times$ of units in F's valuation ring. Let $\mathfrak{m}_F \subset \mathscr{O}_F$ be its maximal ideal, and let $k := \mathscr{O}_F/\mathfrak{m}_F$. There is a short exact sequence

$$1 \longrightarrow 1 + \mathfrak{m}_F \longrightarrow \mathscr{O}_F^\times \longrightarrow k^\times \longrightarrow 0,$$

and $1 + \mathfrak{m}_F$ is pro-p. Similarly, $\mathrm{Aut}(T_\ell(A))$ sits in a short exact sequence

$$1 \longrightarrow 1 + \ell \cdot \mathrm{End}(T_\ell(A)) \longrightarrow \mathrm{Aut}(T_\ell(A)) \longrightarrow \mathrm{Aut}(T_\ell(A)/\ell) \longrightarrow 1,$$

and $1 + \ell \cdot \mathrm{End}(T_\ell(A))$ is pro-ℓ. Comparing these, we see that $1 + \mathfrak{m}_F$ must have finite image in $\mathrm{Aut}(T_\ell(A))$, so $\mathscr{O}_F^\times$ must have finite image in $\mathrm{Aut}(T_\ell(A))$, so inertia must have finite image in $\mathrm{Aut}\, T_\ell(A)$. ∎

Proposition 1.16. *Let (A, i) be an abelian variety of CM-type (E, Φ) over some number field $K \subset \mathbb{C}$. Then, A has potential good reduction over all $\mathfrak{p} \in \mathrm{Spec}\, \mathscr{O}_K$.*

[3] Write $E = \mathbb{Q}(\alpha)$. Then, the eigenvalues of α acting on $T_0(A)$ determine the CM type.

Proof. Fix some $\mathfrak{p} \in \operatorname{Spec} \mathscr{O}_K$, as well as a rational prime $\ell \notin \mathfrak{p}$. Let $G_K = \operatorname{Gal}(K^s/K)$, and consider the ℓ-adic Tate representation

$$\rho_{A,\ell} : G_K \longrightarrow \operatorname{Aut}(V_\ell(A)) \qquad V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Let $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, and note that $V_\ell(A)$ is a free E_ℓ-module of rank 1.[4] Therefore, $\rho_{A,\ell}$ really lands in $\operatorname{Aut}_{E_\ell}(V_\ell(A)) \simeq E_\ell^\times$. In particular, it has abelian image, so we win by Lemma 1.15. ■

2 Interlude: Shimura-Taniyama Formula

We know by Proposition 1.14 that all CM abelian varieties A have a model defined over some number field. We further know, by Proposition 1.16, that we can always choose this number field so that A has everywhere good reduction. Given such a setup, one may ask, "How does Frobenius act on the reductions of A at various primes?"

The main result (Theorem 2.5) of this section will be used in the proof of the Main theorem of complex multiplication. Before stating, we need the following observation.

Notation 2.1. Let $k = \mathbb{F}_q$ be a finite field, and let V/k be a scheme. We let $\operatorname{Fr} = \operatorname{Fr}_V : V \to V$ denote the qth power (absolute) Frobenius map. In particular, Fr is a k-morphism.

Example 2.2. If $V = \mathbb{A}^1_{\mathbb{F}_q}$, $\operatorname{Fr} : V \to V$ is the map $a \mapsto a^q$. △

Lemma 2.3. *Let (A, i) be an abelian variety of CM-type (E, Φ) over a number field $K \subset \mathbb{C}$. Choose a prime $\mathfrak{p} \in \operatorname{Spec} \mathscr{O}_K$ where A has good reduction, say to $(\overline{A}, \bar{i})$ over $\mathbb{F}_q = \mathscr{O}_K/\mathfrak{p}$. Then, the Frobenius map $\operatorname{Fr}_{\overline{A}} \in \operatorname{End}(\overline{A})$ lies in the image of $\bar{i} : E \to \operatorname{End}^0(\overline{A})$.*

Proof. Fix a prime $\ell \neq \operatorname{char} \mathbb{F}_q$, and let $G = \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. It suffices to check the claim after tensoring with $\mathbb{Q}_\ell$.[5] Because

$$\operatorname{End}^0(\overline{A}) \otimes \mathbb{Q}_\ell \longrightarrow \operatorname{End}_G(V_\ell(\overline{A}))$$

is injective[6], it suffices to show that there's some $\alpha \in E$ which acts as Frobenius on $V_\ell(\overline{A})$. We note, as we did in the proof of Proposition 1.16, that $V_\ell(A)$ (so also $V_\ell(\overline{A})$) is a rank one free module over $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Because E_ℓ acts on $V_\ell(A)$ through endomorphisms (up to scaling), this action commutes with that of $\operatorname{Fr}_{\overline{A}}$, so $\operatorname{Fr}_{\overline{A}} \in \operatorname{End}_{E_\ell}(V_\ell(\overline{A})) = E_\ell^\times$, and we win. ■

Notation 2.4. With a setup as in Lemma 2.3, we let $\pi = \pi_{(A,i),\mathfrak{p}} \in E$ denote the element which acts by Frobenius on $\overline{A}$.

[4] e.g. because $V_\ell A \simeq \mathrm{H}_1(A(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell$

[5] Maybe the quickest way to see this is to note that $\mathbb{Q}_\ell$ is faithfully flat over $\mathbb{Q}$, so the composition $E \xrightarrow{\bar{i}} \operatorname{End}^0(A) \twoheadrightarrow \operatorname{End}^0(A)/\operatorname{Fr}_{\overline{A}}$ is zero iff it is after tensoring with $\mathbb{Q}_\ell$.

[6] In fact, bijective by [Tat66]

Theorem 2.5 (Shimura-Taniyama)**.** *Use notation as in Lemma 2.3. Assume that $K/\mathbb{Q}$ is Galois and contains all conjugates of E. Then, for all primes v of E lying over p,*

$$\frac{\mathrm{ord}_v(\pi)}{\mathrm{ord}_v(q)} = \frac{\#(\Phi \cap H_v)}{\#H_v},$$

where $H_v := \left\{\rho : E \to K : \rho^{-1}(\mathfrak{p}) = \mathfrak{p}_v\right\}$.

For a proof, see [Mil07, Section 2] and/or [Mil20, Section 8] (possibly also [Mil17, Section 10]).

Remark 2.6. In proving Theorem 2.5, one usually obtains a stronger result. Under the additional assumption that $i^{-1}(\mathrm{End}(A)) = \mathscr{O}_E$ (used to ensure $\pi \in \mathscr{O}_E$), the ideal $(\pi) \subset \mathscr{O}_E$ factors as

$$(\pi) = \prod_{\varphi \in \Phi} \varphi^{-1}\left(\mathrm{Nm}_{K/\varphi E}\, \mathfrak{p}\right). \qquad \circ$$

3 Main Theorem of Complex Multiplication

Recall, from the introduction, that, given an abelian variety A w/ CM by E, the main theorem is meant to describe the action on A of the Galois group of a so-called associated "reflex field." In what remains, we describe this field and theorem. In addition to [Mil17], this section also borrows from [Mil20] (especially sections II.$\{8, 9\}$).

Definition 3.1. Let (E, Φ) be a CM-type. View Φ as a subset of $\mathrm{Hom}(E, \overline{\mathbb{Q}})$. The associated `reflex field` E^* is the smallest subfield of $\overline{\mathbb{Q}}$ such that there exists an $E \otimes_{\mathbb{Q}} E^*$-module V satisfying

$$V \otimes_{E^*} \overline{\mathbb{Q}} \simeq \bigoplus_{\varphi \in \Phi} \overline{\mathbb{Q}}_\varphi \text{ as } E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}\text{-modules.} \qquad \diamond$$

Remark 3.2. V as above, is uniquely determined up to isomorphism.[7] Furthermore, letting $T = \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m$ and $T^* = \mathrm{Res}_{E^*/\mathbb{Q}}\, \mathbb{G}_m$, there is a homomorphism $N_\Phi : T^* \to T$, called the `reflex norm`, whose action on $\mathbb{Q}$-points is

$$\begin{array}{ccccccc} (E^*)^\times & = & T^*(\mathbb{Q}) & \longrightarrow & T(\mathbb{Q}) & = & E^\times \\ a & & & \longmapsto & & & \det_E(a \mid V) \end{array}$$

(where $\det_E$ denotes determinant on the V viewed as an E-vector space). Note N_Φ also defines a compatible homomorphism on idèles, $\mathbb{A}_{E^*}^\times = T^*(\mathbb{A}_{\mathbb{Q}}^\times) \to T(\mathbb{A}_{\mathbb{Q}}^\times) = \mathbb{A}_E^\times$. $\circ$

Fact (Aside after Definition 11.1, [Mil17])**.** E^* defined above is the fixed field of $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \sigma\Phi = \Phi\}$.

Remark 3.3. If (A, i) is an abelian variety of CM-type (E, Φ), then any field of definition K of (A, i) must contain E^*. This is because $T_0(A)$ has commuting actions of E and K, and a decomposition (over $\mathbb{C}$, so also over $\overline{\mathbb{Q}}$) of the type appearing in Definition 3.1. $\circ$

[7] One can think of it as the E^*-linear representation of $\mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m$ given by $\mathrm{Gal}(\overline{\mathbb{Q}}/E^*)$-stable set of characters Φ.

Before stating the main theorem of complex multiplication, we recall the main theorem of global class field theory.

Theorem 3.4. *Let K be a number field. Then, there is a unique continuous, surjective homomorphism*

$$\mathrm{Art}_K = \varphi_K : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

satisfying

(1) *For any finite place v of K and any finite, abelian extension L/K unramified at v, one has*

$$\varphi_K(\pi_v)|_L = \mathrm{Frob}_v^{-1} \in \mathrm{Gal}(L/K),$$

for any uniformizer π_v of K_v.

(2) *For any finite, abelian extension L/K, φ_K descends to an isomorphism*

$$\varphi_{L/K} : K^\times \backslash \mathbb{A}_K^\times / \mathrm{Nm}_{L/K}(\mathbb{A}_L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

We write $\varphi_K : \mathbb{A}_K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ also for its lift to $\mathbb{A}_K^\times$.

Remark 3.5. The normalization in **(1)** above is the inverse of the usual one, but is chosen here to ensure that

$$\varphi_{\mathbb{Q}}(\chi(\sigma)) = \sigma,$$

where $\chi : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times \subset \mathbb{A}_f^\times$ is the usual cyclotomic character. ◦

Remark 3.6. Let (A, i) be an abelian variety with CM type (E, Φ), $\Phi \subset \mathrm{Hom}(E, \overline{\mathbb{Q}})$. Choose any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E^*)$, so $\sigma\Phi = \Phi$. Hence, A^σ has CM type (E, Φ) as well, so (Proposition 1.13) there is an E-isogeny $\alpha : A \to A^\sigma$. Let $\mathbb{A}_{f,E} := E \otimes \mathbb{A}_f$. Both maps

$$x \mapsto \sigma x, \;\; x \mapsto \alpha x : V_f(A) \longrightarrow V_f(A^\sigma)$$

are $\mathbb{A}_{f,E}$-linear isomorphisms. Because $V_f(A)$ is free of rank one, they must differ by some $\eta(\sigma) \in \mathbb{A}_{f,E}^\times$, i.e.

$$\alpha(\eta(\sigma)x) = \sigma x \;\text{ for all }\; x \in V_f(A).$$

Changing α only changes $\eta(\sigma)$ by an element of $E^\times$, so we get a well-defined map

$$\eta : \mathrm{Gal}(\overline{\mathbb{Q}}/E^*) \longrightarrow \mathbb{A}_{f,E}^\times / E^\times. \qquad ◦$$

The main theorem says that η above is the reflex norm.

Theorem 3.7 (`Main Theorem of Complex Multiplication`)**.** *Let (A, i) be an abelian variety with CM type (E, Φ) over $\overline{\mathbb{Q}}$, and let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E^*)$. For any $s \in \mathbb{A}_{f,E}^\times$ with $\varphi_{E^*}(s) = \sigma|_{(E^*)^{\mathrm{ab}}}$, there is a unique E-"isogeny" $\alpha : A \to A^\sigma$ (i.e. $\alpha \in \mathrm{Hom}^0(A, A^\sigma) = \mathrm{Hom}(A, A^\sigma) \otimes \mathbb{Q}$) such that*

$$\alpha(N_\Phi(s) \cdot x) = \sigma x \;\; \textit{for all} \;\; x \in V_f(A).$$

Remark 3.8. One can replace $\mathrm{Gal}(\overline{\mathbb{Q}}/E^*)$ with $\mathrm{Aut}(\mathbb{C}/E^*)$ in the above theorem. This follows from Proposition 1.14. ◦

We will not prove this here[8]. However, one can find a proof in [Mil07, Section 3] or [Mil20, Section 8]. It is claimed in [Mil17, Section 11] that the Shimura-Taniyama formula can be used to show that η is given by the reflex norm, but I don't see how. However, we will end by connecting this theorem to moduli (recall our initial motivation of producing canonical models of Shimura varieties).

3.1 Relation to moduli

Let (E, Φ) be a CM-type, and let $F \subset E$ be E's maximal totally real subfield. Consider the tori $T^E := \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m$ and $T^F := \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m$. Let $\mathrm{Nm} : T^E \to T^F$ denote the norm map $a \mapsto a \cdot \overline{a}$. Let T be the fiber product

$$\begin{array}{ccc} T & \longrightarrow & T^E \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Nm}} \\ \mathbb{G}_m & \longrightarrow & T^F, \end{array}$$

i.e. $T(R) = \left\{ a \in (E \otimes_{\mathbb{Q}} R)^\times : \varphi_R(a)\overline{\varphi_R(a)} = \varphi'_R(a)\overline{\varphi'_R(a)} \in \mathbb{C} \otimes_{\mathbb{Q}} R \text{ for any } \varphi, \varphi' \in \Phi \right\}$.

Consider the morphism $h_\Phi : \mathbb{S} \to T^E_{\mathbb{R}}$ defined on $\mathbb{R}$-points by

$$\begin{array}{rccc} h_\Phi(\mathbb{R}) : & \mathbb{C}^\times & \longrightarrow & E^\times_{\mathbb{R}} \simeq \prod_{\varphi \in \Phi} \mathbb{C}_\varphi \\ & z & \longmapsto & (z, z, \ldots, z). \end{array}$$

Lemma 3.9. *The image of h_Φ lands in $T_{\mathbb{R}} \subset T^E_{\mathbb{R}}$, and the pair $(T, \{h_\Phi\})$ is a Shimura datum.*

Proof. For the first part, simply compute that the composition

$$\mathbb{S} \xrightarrow{h_\Phi} T^E_{\mathbb{R}} \xrightarrow{\mathrm{Nm}} T^F_{\mathbb{R}} \simeq \prod_{\varphi \in \Phi} \mathbb{R}_\varphi$$

is $z \longmapsto (|z|^2, \ldots, |z|^2)$, so $\mathrm{im}\, h_\Phi \subset \mathrm{Nm}^{-1}(\mathbb{G}_m) = T$. Now, conditions **(SV1,2,3)** are vacuous since T is a torus (so commutative and noncompact). ■

We thus get a Shimura variety $\mathrm{Sh}(T, \{h_\Phi\})$, which we claim is a moduli space for abelian varieties with CM.

Construction 3.10. Fix a purely imaginary $\alpha \in \mathscr{O}_E$ such that $\mathrm{Im}\,\varphi(\alpha) > 0$ for all $\varphi \in \Phi$, as in the proof of Proposition 1.13. Let $\psi : E \times E \to \mathbb{Q}$ be the bilinear form

$$\psi(x, y) := \mathrm{Tr}_{E/\mathbb{Q}}(\alpha x \overline{y}) = \sum_{\varphi \in \Phi} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\left(\varphi(\alpha)\varphi(x)\overline{\varphi(y)}\right).$$

[8] mostly because I do not understand the proof(s)

With this defined, $T \subset T^E$ can be described as the torus

$$T(R) := \left\{(a,b) \in (E \otimes_{\mathbb{Q}} R)^{\times} \times R^{\times} : \psi(ax, ay) = b\psi(x,y) \text{ for all } x, y \in E \otimes_{\mathbb{Q}} R.\right\}. \tag{3.1}$$

Note above that $b = \varphi_R(a)\overline{\varphi_R(a)}$ for any $\varphi \in \Phi$. ○

Definition 3.11. For any compact open subgroup $K \subset T(\mathbb{A}_f)$, let $\mathcal{M}_K$ denote the set of isomorphism classes of quadruples $(A, j, \lambda, \eta K)$ where

- A is a complex abelian variety
- λ is a polarization, equivalently a Riemann form $\psi_\lambda : \mathrm{H}_1(A, \mathbb{Q}) \times \mathrm{H}_1(A, \mathbb{Q}) \to \mathbb{Q}$
- j is a homomorphism $E \to \mathrm{End}^0(A)$
- ηK is a K-orbit of $\mathbb{A}_{E,f}$-linear (note $\mathbb{A}_{E,f} = E \otimes_{\mathbb{Q}} \mathbb{A}_f$) isomorphisms $\eta : \mathbb{A}_{E,f} \to V_f(A)$,

and which satisfy

$$\text{there exists an } E\text{-linear isomorphism } a : \mathrm{H}_1(A, \mathbb{Q}) \to E \text{ sending } \psi_\lambda \text{ to a } \mathbb{Q}^{\times}\text{-multiple of } \psi. \tag{$\star$}$$

An isomorphism from one tuple $(A, i, \eta K)$ to another $(A', i', \eta' K)$ is an E-"isogeny" $A \to A'$ sending η to η' modulo K. ◇

Proposition 3.12. *For any compact open subgroup $K \subset T(\mathbb{A}_f)$, there is a natural bijection*

$$\mathcal{M}_K \xrightarrow{\sim} \mathrm{Sh}_K(T, \{h_\Phi\}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K.$$

Proof Sketch. Say $(A, j, \lambda, \eta K) \in \mathcal{M}_K$. Fix any choice of isomorphism $a : \mathrm{H}_1(A, \mathbb{Q}) \xrightarrow{\sim} E$ as in ($\star$). Using that $V_f(A) = \mathrm{H}_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f$, we can consider the composition

$$E \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\eta} V_f(A) = \mathrm{H}_1(A, \mathbb{Q}) \otimes \mathbb{A}_f \xrightarrow{a} E \otimes \mathbb{A}_f.$$

By definition, this satisfies (3.1), so $a \circ \eta \in T(\mathbb{A}_f)$. The choice of a is determined up to composition w/ an element of $T(\mathbb{Q})$, and similarly η is determined up to composition with an element of K. Therefore, the class of $a \circ \eta$ in $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$ is well-defined. The desired map is the just-described $(A, j, \lambda, \eta K) \mapsto [a \circ \eta]$. ■

Theorem 3.13. *Let $\mathrm{Aut}(\mathbb{C}/E^*)$ act on $\mathrm{Sh}_K(T, \{h_\Phi\}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$ as follows: given $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$, choose $s \in \mathbb{A}_{f,E^*}^{\times}$ such that $\mathrm{Art}_{E^*}(s) = \sigma|_{(E^*)^{\mathrm{ab}}}$, and then set*

$$\sigma[a] := [N_\Phi(s) \cdot a] \quad \textit{for } a \in T(\mathbb{A}_f).$$

The bijection

$$\mathcal{M}_K \xrightarrow{\sim} \mathrm{Sh}_K(T, \{h_\Phi\})$$

is equivariant for this action.

Proof idea. Let $\alpha : A \to A^\sigma$ be the E-"isogeny" of Theorem 3.7. By Theorem 3.7 (+ [Mil20, Remark 9.11(c)] to verify $(\star)$), if $(A, j, \lambda, \eta K) \in \mathcal{M}_K$ corresponds to $[a \circ \eta] \in T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$, then $(A^\sigma, j^\sigma, \lambda^\sigma, \eta^\sigma K)$ corresponds to $[b \circ \eta^\sigma]$, where $b = a \circ V_f(\alpha)^{-1}$. Expanded out, this is

$$b \circ \eta^\sigma = a \circ V_f(\alpha)^{-1} \circ \sigma \circ \eta = a \circ N_\Phi(s) \circ \eta = N_\Phi(s) a \circ \eta,$$

where $s \in \mathbb{A}_{f,E}^\times$ is chosen so that $\varphi_{E^*}(s) = \sigma|_{(E^*)^{\mathrm{ab}}}$. This proves the claim. ■

Appendices

Note 1. I waited sufficiently long after the end of the semester before writing the appendices that I have forgotten what all I originally wanted to put in here...

A Review of Shimura Data

Let's first define (connected) Shimura data.

Notation A.1. We will often make reference to the Deligne torus $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ as well as the $\mathbb{R}$-algebraic group U_1 defined by

$$U_1(R) := \left\{ g \in (R \otimes_{\mathbb{R}} \mathbb{C})^{\times} : g\bar{g} = 1 \right\},$$

where the bar $^{-}$ denotes complex conjugation. Note that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ while $U_1(\mathbb{R}) = S^1$ is the circle group.

Definition A.2. A pair (G, D) consisting of a semisimple $\mathbb{Q}$-algebraic group G and a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class D of homomorphisms $u : U_1 \to G^{\mathrm{ad}}_{\mathbb{R}}$ is called a connected Shimura datum if it satisfies all of

- **(SU1)** For all $u \in D$, the only characters appearing in the representation $\mathrm{Ad} \circ u$ of U_1 on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ are $z^{-1}, 1, z$.
- **(SU2)** For all $u \in D$, $\mathrm{ad} u(-1)$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.
- **(SU3)** G^{ad} has no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact. ◇

For 'Shimura data,' there are additional axioms which one can consider. All the ones appearing in [Mil17, Section 5] are listed below.

Definition A.3. Let $G/\mathbb{Q}$ be a connected[9], reductive algebraic group, and let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$. We consider the following axioms:

- **(SV1)** For each $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$.

  When this is satisfied, one gets a canonically attached weight homomorphism $w_X : \mathbb{G}_{m,\mathbb{R}} \longrightarrow Z(G)^{\circ}_{\mathbb{R}} \subset G_{\mathbb{R}}$.
- **(SV2)** For each $h \in X$, $\mathrm{ad} h(i)$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.
- **(SV2*)** For each $h \in X$, $\mathrm{ad} h(i)$ is a Cartan involution on $G_{\mathbb{R}}/w_X(\mathbb{G}_m)$.
- **(SV3)** G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

[9] Milne [Mil17] notes in several places that much of the theory goes through even without assuming G is connected.

(SV4) The weight homomorphism $w_X : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$ (one says "`the weight is rational`").

(SV5) The group $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$.

(SV6) The torus Z° splits over a CM field.

We call the pair (G, X) a `Shimura datum` if it satisfies **(SV1)**,**(SV2)**,**(SV3)**. $\diamond$

Remark A.4. Because of technicalities in the definition of Shimura datum, there's also an ever-so-slightly separate notion of a "*zero-dimensional Shimura datum/variety*" (see e.g. [Mil17, Towards end of chapter 5]). I won't bother recalling this here, but in short, when $G = T$ is a torus, it is useful to allow X to be bigger than a conjugacy class (= singleton set).[10] $\circ$

Given a Shimura datum (G, X) and a compact open $K \subset G(\mathbb{A}_f)$, the corresponding Shimura variety is given by

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \cong G(\mathbb{A}) \times G(\mathbb{A})/(K_\infty \times K),$$

where $K_\infty \subset G(\mathbb{R})$ is the centralizer of h.

[10] Actually, Milne [Mil17, End of Chapter 9] remark that it might be better to loosen the definition of Shimura datum by allowing X to be a finite cover of a $G(\mathbb{R})$-conjugacy class.

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