Local Heights Notes*

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Contents

Introduction

Our goal in this seminar is to prove the following theorem.

Theorem 1 (**Gross-Zagier Formula**, roughly stated)**.** *Let f be a newform for $\Gamma_0(N)$, let $X := X_0(N)$, and let $J = \mathrm{Jac}(X)$. Let K be an imaginary quadratic field, with Hilbert class field H, supporting a Heegner point $x \in X(H)$. Then, for each character $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$, there exists some $c_{f,\chi} \in J(H)$ such that*

$$L'(f, \chi, 1) = \frac{8\pi^2(f, f)}{[H : K]\sqrt{D}}\widehat{h}_{2\Theta}(c_{\chi,f}),$$

where $\widehat{h}_{2\Theta}$ denotes the Néron-Tate canonical height associated to the divisor 2Θ.

The rough strategy to do so is simply to compute both sides of the claimed equality, and see that they come out to be the same thing. On the right hand side, we have this (global) canonical height on the Jacobian of $X_0(N)$. In order to compute this, we will want to express it as a sum

$$\sum_v h_{2\Theta,v} = \widehat{h}_{2\Theta}$$

of local heights attached to each place v of H. Our goal in the current talk/notes is to do just that. Specifically, we want the following:

*Including my confusions

Let K be a global field, and let (C, P_0) be a 'nice' pointed K-curve (in particular, $P_0 \in C(K)$) of genus $g \geq 1$. Let $J = \mathrm{Jac}(C)$, and let $j : C \hookrightarrow J$ be the Abel-Jacobi map $P \mapsto [P] - [P_0]$. Let

$$\Theta := \underbrace{j(C) + \cdots + j(C)}_{(g-1) \text{ summands}} \subset J$$

be the Theta divisor on J, and let $\widehat{h} = \widehat{h}_{2\Theta}$ be the Néron-Tate height associated to 2Θ.

Fact. Θ is an ample divisor.

Remark 2. Let $\Theta^- := [-1]^*\Theta$. We note the following

- $\Theta + \Theta^-$ is an ample, symmetric divisor. Hence, $\widehat{h}_{\Theta+\Theta^-} : J(K) \otimes \mathbb{R} \to \mathbb{R}$ is a positive-definite quadratic form.
- $\Theta - \Theta^-$ is a (non-ample) anti-symmetric divisor. Hence, $\widehat{h}_{\Theta-\Theta^-} : J(K) \otimes \mathbb{R} \to \mathbb{R}$ is a linear form.
- $2\Theta = (\Theta + \Theta^-) + (\Theta - \Theta^-)$, so $\widehat{h}_{2\Theta} = \widehat{h}_{\Theta+\Theta^-} + \widehat{h}_{\Theta-\Theta^-}$ is a quadratic function with associated bilinear form

  $$\langle \alpha, \beta \rangle_{2\Theta} = \frac{1}{2}\left(\widehat{h}_{2\Theta}(\alpha+\beta) - \widehat{h}_{2\Theta}(\alpha) - \widehat{h}_{2\Theta}(\beta)\right) = \frac{1}{2}\left(\widehat{h}_{\Theta+\Theta^-}(\alpha+\beta) - \widehat{h}_{\Theta+\Theta^-}(\alpha) - \widehat{h}_{\Theta+\Theta^-}(\beta)\right) = \langle \alpha, \beta \rangle_{\Theta+\Theta^-}$$

  on $(J(K) \otimes \mathbb{R}) \times (J(K) \otimes \mathbb{R})$. ○

Theorem 3. *With notation as above, for each place v of K, there is a (partially defined) symmetric, bi-additive function*

$$\langle -, - \rangle_v : \mathrm{Div}^0(C) \times \mathrm{Div}^0(C) \dashrightarrow \mathbb{R}$$

such that for $a, b \in \mathrm{Div}^0(C)$, one has

$$\sum_v \langle a, b \rangle_v = \langle a, b \rangle_{2\Theta}.$$

In particular,

$$\sum_v \langle a, a \rangle_v = \widehat{h}_{2\Theta}(a).$$

Remark 4. Actually, the $\langle -, - \rangle_v$'s can be extended to functions on all of $\mathrm{Div}^0(C) \times \mathrm{Div}^0(C)$ so that the above decomposition still holds, see Remark 1.8 and [Gro86, Section 5]. ○

With our goal stated, the first things to do are to definite these local pairing precisely, and then construct them.

Note 1. Our main references for all that follows are [Gro86, Sections 1–5], [Sil94, Section IV.7], and [Sil].

1 Setup and Definition of Local Height Pairing

Before precisely defining the local height pairing, we must setup some notation/conventions.

Definition 1.1. Let K be a field. We'll call a K-scheme X/K **nice** if it is smooth, projective, and geometrically connected (= geometrically integral since smooth). ◇

Definition 1.2. Let X be a scheme. Two (Weil) divisors $a, b \in \mathrm{Div}(X)$ are said to be **relatively prime** if they share no irreducible components. $\diamond$

Setup 1.3. We maintain the following conventions/notation

- $K = K_v$ is a local field (with valuation ring $\mathscr{O}_K = \{v(x) \geq 0\}$ if v non-arch).
- C/K is a nice K-curve such that $C(K) \neq \emptyset$.
- $J = \mathrm{Pic}^0_{C/K}$ is the Jacobian of C.
- $Z^0(C/K)$ denotes the degree zero elements in the free group on $C(K)$

The local height pairing will be defined on (certain) pairs of (degree 0) divisors on C, so let us introduce some relevant notation here.

Notation 1.4. Let $f \in K(C)^\times$ be a rational function. Let $a = \sum_P n_P[P] \in \mathrm{Div}^0(X)$ be a divisor supported away from $\mathrm{div}(f)$. We set

$$f(a) := \prod_P f(P)^{n_P}.$$

Note that $f(a) = 1$ if $f \in K^\times$ is constant, so $f(a)$ depends only the divisor $\mathrm{div}(f)$ of f.

The last thing we will need is to fix a choice $|\cdot|_K : K^\times \to \mathbb{R}^\times_{>0}$ of absolute value on K. To do this, we let $\mu = \mu_K$ denote any additive Haar measure on K. For $\alpha \in K$, consider the Haar measure $\mu_\alpha(E) := \mu(\alpha E)$ (where $E \subset K$ a Borel set), and we define $|\alpha|_K$ via the equality

$$\mu_\alpha = |\alpha|_K \, \mu,$$

so $|\cdot|_K$ is well-defined by uniqueness of Haar measures up to scaling.

Example. Say K is a non-archimedean local field, with residue field $\mathbb{F}$ of size q. Let $\pi \in \mathscr{O}_K$ be a uniformizer. Then, $\mathscr{O}_K = \bigsqcup_{\overline{x} \in \mathbb{F}} (\pi\mathscr{O}_K + x)$ and $\mu(\pi\mathscr{O}_K + x) = \mu(\pi\mathscr{O}_K)$ since μ is an additive Haar measure, so

$$\mu(\mathscr{O}_K) = \sum_{\overline{x} \in \mathbb{F}} \mu(\pi\mathscr{O}_K + x) = q\mu(\pi\mathscr{O}_K) = q\mu_\pi(\mathscr{O}_K) \implies |\pi|_K = \frac{1}{q}.$$

In general, one gets $|\alpha|_K = q^{-v(\alpha)}$, where $v : K \to \mathbb{Z}$ is the valuation on K (normalized by $v(\pi) = 1$). $\triangle$

With all this notation introduced, our first big goal is the existence and uniqueness of (something like) a symmetric, bilinear pairing $\mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \to \mathbb{R}$. The actual pairing we obtain will not be defined on all pairs of inputs, but only on those given by relatively prime divisors. Hence, we introduce the following non-standard notation

Notation 1.5. Let $A, B \subset \mathrm{Div}(X)$ be two subgroups. For a set Y, we will write $f : A \times B \overset{\mathrm{RP}}{\dashrightarrow} Y$ to denote a function f defined on the subset

$$\{(a, b) \in A \times B : a, b \text{ are relatively prime}\} \subset A \times B.$$

Theorem 1.6 ([Nér65], Theéorèm II.9.3)**.** *There is a unique function*

$$\langle -,-\rangle_K : Z^0(C/K) \times \mathrm{Div}^0(C) \overset{RP}{\dashrightarrow} \mathbb{R}$$

satisfying

(1) $\langle a,b\rangle_K + \langle a,c\rangle_K = \langle a, b+c\rangle_K$

(2) $\langle a,b\rangle_K = \langle b,a\rangle_K$ *when* $b \in Z^0(X/K)$

(3) $\langle a, \mathrm{div}(f)\rangle_K = \log|f(a)|_K$.
When K *non-arch, this is* $-v(f(a))\log q$.

(4) Fix $b \in \mathrm{Div}^0(C)$ *and a point* $x_0 \in C(K_v) \setminus \operatorname{supp} b$*. Then, the function*

$$C(K_v) \setminus \operatorname{supp} b \to \mathbb{R}, \ \ x \mapsto \langle [x]-[x_0], b\rangle_K$$

is continuous.

Remark 1.7. If H is a finite extension of $K = K_v$ and a, b are relatively prime divisors of degree zero over K, with $a \in Z^0(C/K_v)$, we have

$$\langle a,b\rangle_H = [H:K]\langle a,b\rangle_K .$$

This follows from uniqueness once we know $(a,b) \mapsto \langle a,b\rangle_H / [H:K]$ satisfies **(3)** (the rest are easy).

Say $f \in K(C)^\times$ and $a \in Z^0(C/K_v)$. Then, (if v non-arch)

$$|f(a)|_H = q_H^{-v_H(f(a))} = \left(q_K^{f(H|K)}\right)^{-v_K(f(a))e(H|K)} = \left(q_K^{-v_K(f(a))}\right)^{[H:K]}$$

from which the claim follows.

Hence, we can extend $\langle a,b\rangle_v$ to a pairing between relatively prime divisors in $\mathrm{Div}^0(C)$; choose a finite extension H where a becomes pointwise rational, and then set

$$\langle a,b\rangle_K := \frac{1}{[H:K]}\langle a,b\rangle_H .$$

In this way, Theorem 1.6 really gives a unique pairing

$$\langle -,-\rangle_K : \mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \overset{\mathrm{RP}}{\dashrightarrow} \mathbb{R}$$

(which can be defined even if X has no rational points) ○

Remark 1.8. It is possible to extend this pairing to one defined on divisors with common support. First, at each point $x \in C(K)$, choose a basis $(\partial/\partial t)_x$ for the tangent space at x along with a uniformizer z_x so that $(\partial/\partial t)_x(z_x) = 1$. Now, for any $f \in K(C)^\times$, we set

$$f[x] := \left(\frac{f}{z_x^{\mathrm{ord}_x(f)}}\right)(x) \in K^\times.$$

We similarly define $f[a]$ for $a \in Z^0(C/K)$. Now, to pair $a \in Z^0(C/K)$ with $b \in \mathrm{Div}^0(C)$, we choose a function $f \in K(C)^\times$ s.t. $b - \mathrm{div}(f) =: b'$ is relatively prime to a, and then we set

$$\langle a, b\rangle_K := \log |f[a]|_K + \langle a, b'\rangle_K .$$

This pairing is dependent only on the choice of tangent vectors $(\partial/\partial t)_x$ (and only on them up to scaling by a unit $u \in \mathscr{O}_K^\times$). $\circ$

Definition 1.9. A **correspondence** $T : X \dashrightarrow Y$ is a diagram[1]

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

with f, g both finite maps of curves. $\diamond$

Example (not stated carefully)**.** Fix a (squarefree) number N, and a prime $p \nmid N$. Recall that $X_0(N)$ is the coarse moduli scheme for the functor/stack parameterizing (generalized) elliptic curves equipped with a cyclic subgroup of order N. Note that, since $p \nmid N$, we can think of $X_0(Np)$ as parameterizing pairs $(E \xrightarrow{\varphi} E', G)$ of a (cyclic) p-isogeny and a cyclic order N subgroup $G \subset E$; in effect, p- and N-isogenies are distinct. With this in mind, one obtains a **Hecke correspondence**

$$\begin{tikzcd} & X_0(Np) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_0(N) & & X_0(N), \end{tikzcd}$$

where

$$f : (E \xrightarrow{\varphi} E', G) \mapsto (E, G) \text{ and } g : (E \xrightarrow{\varphi} E', G) \mapsto (E', \varphi(G)). \qquad \triangle$$

Fact. For $a \in \mathrm{Div}^0(X)$ and $b \in \mathrm{Div}^0(Y)$, one has[2]

$$\langle a, T^* b\rangle_X = \langle T_* a, b\rangle_Y$$

whenever both sides are defined.

We still need to prove Theorem 1.6. The uniqueness part of the theorem is maybe easy. Indeed, if $\langle -,-\rangle_K$ and $\langle -,-\rangle_K'$ both satisfy the theorem statement, then for any fixed $a \in Z^0(C/K)$, one obtains a continuous homomorphism

$$\begin{array}{ccc} \mathrm{Jac}(C)(K) & \longrightarrow & \mathbb{R} \\ \beta & \longmapsto & \langle a, \beta\rangle_K - \langle a, \beta\rangle_K' \end{array}$$

This is necessarily trivial since $\mathrm{Jac}(C)(K)$ is a compact topological space (with this analytic topology) and $\mathbb{R}$ has no nontrivial compact subgroups (consider multiples of an element with non-trivial absolute value).

[1] The "correct" definition might actually be to say that it's a divisor on $X \times Y$ (at least when X, Y are both curves), but I think what I say here is good enough for us.
[2] $T^* = f_* g^*$ while $T_* = g_* f^*$

Warning 1.10. It's not obvious to me that the morphism defined above is continuous. Presumably this follows from **(4)** + **(2)** of Theorem 1.6, but I don't see how to show this... The argument above is (more-or-less) what's stated in [Gro86] and [Sil], but [Nér65] gave a different argument that is less immediate. There's another proof in [Lan83, Lemma 11.2.3] which I suspect is "the same" as the one Néron gave, but I have not read either of these closely. •

The existence part of Theorem 1.6 is harder. In these notes, we will prove it only for non-archimedean K, where the desired pairing can be constructed via intersection theory for relative curves over $\mathscr{O}_K$. With this in mind, we first take a detour to describe the main points of the relevant intersection theory.

2 Intersection Theory on Arithmetic Surfaces

Setup 2.1. Let R be a dvr with maximal ideal $\mathfrak{p} = (\pi)$, residue field $k = R/\mathfrak{p}$, and fraction field K.

Definition 2.2. An **arithmetic surface** over R is an integral, normal, excellent, flat, finite type R-scheme $\mathcal{C}/R$ whose generic fiber is a nice K-curve. ◇

Don't ask me what excellent means

Lemma 2.3. *Let $\mathcal{C}/R$ be a proper arithmetic surface. Then, its special fiber $\mathcal{C}_k$ is connected.*

Proof. Let $\pi : \mathcal{C} \to \operatorname{Spec} R$ denote the structure map, and consider the Stein factorization

$$\mathcal{C} \xrightarrow{f} \mathbf{Spec}_R(\pi_*\mathscr{O}_{\mathcal{C}}) = \operatorname{Spec}\Gamma(\mathcal{C}, \mathscr{O}_{\mathcal{C}}) \xrightarrow{g} \operatorname{Spec} R,$$

so f has connected fibers, and g is finite. Since the generic fiber $\mathcal{C}_K$ is proper and geometrically connected, $\Gamma(\mathcal{C}, \mathscr{O}_{\mathcal{C}})$ is generically of rank 1 over R, and so since $\mathcal{C}$ is reduced, $\Gamma(\mathcal{C}, \mathscr{O}_{\mathcal{C}}) = R$. Thus g is the identity, so $\pi = f$ has connected fibers. ■

The start of intersection theory on such surfaces is the following

Definition 2.4. Le $D_1, D_2 \in \operatorname{Div}(\mathcal{C})$ be distinct, effective irreducible divisors, and let $x \in \mathcal{C}$ be a closed point in the special fiber $\mathcal{C}_k$ of $\mathcal{C}$. Choose uniformizers $f_1, f_2 \in \mathscr{O}_{\mathcal{C},x}$ for D_1, D_2. The **intersection index of D_1, D_2 at x** is

$$(D_1 \cdot D_2)_x := \operatorname{length}_R \mathscr{O}_{\mathcal{C},x}/(f_1, f_2). \qquad \diamond$$

Remark 2.5. With notation as above, if x is a regular point of D_1, then one has

$$(D_1 \cdot D_2)_x = \operatorname{ord}_{D_1}(\overline{f}_2),$$

where ord_{D_1} is the valuation on $\mathscr{O}_{D_1,x} = \mathscr{O}_{\mathcal{C},x}/(f_1)$. ○

One would hope that adding up these local intersection indices gives a well-behaved global intersection number. In particular, one would want that the intersection number between two divisors depends only on their linear equivalence classes.

Example ([Sil94], Example IV.7.1)**.** Let $\mathcal{C} = \mathbb{P}^1_R = \operatorname{Proj} R[X, Y]$, and consider the divisors

$$D_1 = \{X = 0\} \text{ and } D_2 = \{X + \pi^n Y = 0\}.$$

Their intersection consists only of the point $x = \{X = 0 = \pi\}$ on the special fiber. The local intersection index at this point is

$$(D_1 \cdot D_2)_x = \dim_k \frac{R[X]_{(X)}}{(X, X + \pi^n)} = n.$$

At the same time $D_2 \sim D_3$, where

$$D_3 = D_2 + \operatorname{div}\left(\frac{Y}{X + \pi^n Y}\right) = \{Y = 0\},$$

but D_1, D_3 have no points of intersection. △

To get around this, one restricts the intersection pairing to only certain pairs of divisors; specifically, one only allows intersections where one of the divisors involved is 'vertical'.

Definition 2.6. Let $\mathcal{C}/R$ be an arithmetic surface. A divisor $D \subset \mathcal{C}$ is **vertical** if it is supported in the special fiber. The subgroup of vertical divisors will be denote $\operatorname{VDiv}(\mathcal{C})$. ◇

Note that there is an exact sequence

$$0 \longrightarrow \operatorname{VDiv}(\mathcal{C}) \longrightarrow \operatorname{Div}(\mathcal{C}) \longrightarrow \operatorname{Div}(\mathcal{C}_K) \longrightarrow 0.$$

Theorem 2.7. *Let $\mathcal{C}/R$ be a regular, proper arithmetic surface. There is a unique bilinear pairing*

$$\operatorname{Div}(\mathcal{C}) \times \operatorname{VDiv}(\mathcal{C}) \longrightarrow \mathbb{Z}, \;\; (D, F) \mapsto D \cdot F$$

satisfying

(i) *If $D \in \operatorname{Div}(\mathcal{C})$ and $F \in \operatorname{VDiv}(\mathcal{C})$ are distinct irreducible divisors, then*

$$D \cdot F = \sum_{x \in D \cap F} (D \cdot F)_x.$$

(ii) *If $D_1, D_2 \in \operatorname{Div}(\mathcal{C})$ and $F \in \operatorname{VDiv}(\mathcal{C})$ with $D_1 \sim D_2$, then*

$$D_1 \cdot F = D_2 \cdot F.$$

(iii) *If $F_1, F_2 \in \operatorname{VDiv}(\mathcal{C})$, then*

$$F_1 \cdot F_2 = F_2 \cdot F_1.$$

This pairing furthermore satisfies

(iv) *If E is a prime divisor (= integral, codim 1 subscheme), then*

$$E \cdot F = \deg(\mathcal{O}_{\mathcal{C}}(E)|_F)$$

for any $F \in \operatorname{VDiv}(\mathcal{C})$.

(See [Lic68], but maybe also [Sil94, Theorem 7.2] and [Rom13])

Abuse of Notation 2.8. For any relatively prime divisors $D, E \in \mathrm{Div}(\mathcal{C})$, we set

$$D \cdot E := \sum_{x \in D \cap F} (D \cdot E)_x$$

(even if neither is vertical). Be warned that this intersection product does not enjoy all the properties of Theorem 2.7 in general, e.g. $D \cdot E$ can be nonzero even if D or E is principal (unless the other is vertical).

There are more properties of the intersection pairing we will need. In English, it's restriction to the special fiber gives a negative semi-definite bilinear form with 1-dimensional kernel spanned by the full special fiber $\mathcal{C}_k$. In math,

Theorem 2.9. *Let $\mathcal{C}/R$ be a regular, proper arithmetic surface. For any $F \in \mathrm{VDiv}(\mathcal{C})$, one has $F^2 \le 0$ and the following are equivalent*

(i) $F^2 = 0$

(ii) $F \cdot F' = 0$ *for every* $F' \in \mathrm{VDiv}(\mathcal{C})$

(iii) $F = a\mathcal{C}_k$ *for some* $a \in \mathbb{Q}$

Proof. **(iii)** $\implies$ **(ii)** $\implies$ **(i)** are easy (using that $\mathcal{C}_k = \mathrm{div}(\pi)$ is principal). To see that $F^2 \le 0$, we first write

$$\mathcal{C}_k = \sum_{i=1}^r a_i F_i \text{ and } F = \sum_{i=1}^r c_i F_i = \sum_{i=1}^r \frac{c_i}{a_i}(a_i F_i).$$

Consider the $\mathbb{Q}$-divisor $F' = \sum_{i=1}^r (c_i/a_i)^2(a_i F_i)$. We have $F' \cdot \mathcal{C}_k = 0$, so

$$\begin{aligned}
-2F^2 &= 2F' \cdot \mathcal{C}_k - 2F^2 \\
&= \sum_{i,j=1}^r \left(\frac{c_i^2}{a_i^2}(a_iF_i \cdot a_jF_j) + \frac{c_j^2}{a_j^2}(a_iF_i \cdot a_jF_j) \right) - 2\sum_{i,j=1}^r \frac{c_ic_j}{a_ia_j}(a_iF_i \cdot a_jF_j) \\
&= \sum_{i,j=1}^r \left(\frac{c_i}{a_i} - \frac{c_j}{a_j} \right)^2 (a_iF_i \cdot a_jF_j) \\
&\ge 0.
\end{aligned}$$

This just leaves showing **(i)** $\implies$ **(iii)**. Say $F^2 = 0$. Staring at the above expression, we conclude that

$$F_i \cdot F_j > 0 \implies \frac{c_i}{a_i} = \frac{c_j}{a_j}.$$

Since $\mathcal{C}_k$ is connected (by Remark **??**), we can conclude that there is some $\alpha \in \mathbb{Q}$ with $c_i/a_i = \alpha$ for all i. Thus, $F = \alpha\mathcal{C}_k$. ∎

This should cover all the intersection theory we'll need.

3 Construction of Local Heights in the non-arch Case

We first recall the setup of Theorem 1.6.

Setup 3.1. Let $K = K_v$ be a non-arch local field with valuation ring $\mathscr{O}_K$, uniformizer π, and residue field k. Let C/K be a nice curve with $C(K) \neq \emptyset$.

We wish to define a pairing

$$\langle -, - \rangle_v : Z^0(C/K) \times \mathrm{Div}^0(C) \overset{\mathrm{RP}}{\dashrightarrow} \mathbb{R}$$

satisfying **(1)**–**(4)** of Theorem 1.6. To this then, we first let $\mathcal{C}/\mathscr{O}_k$ be a proper, regular model of C. In order to define a paring between divisors on $C \cong \mathcal{C}_K$ in terms of intersection theory on $\mathcal{C}$, we will need a good way to extend divisor on the generic fiber of $\mathcal{C}$ to the whole of $\mathcal{C}$.

Let $\{F_i\}_{i=1}^r$ be the (reduced) irreducible components of the special fiber $\mathcal{C}_k$.

Lemma 3.2. *A divisor $b \in \mathrm{Div}^0(C)$ extends to a rational divisor $B \in \mathrm{Div}(\mathcal{C}) \otimes \mathbb{Q}$ which satisfies*

$$B \cdot F_i = 0 \text{ for all } i.$$

Furthermore, B is unique up to additions of multiplies of $\mathcal{C}_k$.

Proof. First write $b = \sum_j n_j[p_j]$ with p_j closed in the generic fiber. Let $D_j = \overline{p_j} \subset \mathcal{C}$ be the closure of p_j in $\mathcal{C}$, and set $B' := \sum_j n_j D_j$. Evidently, B' is a ($\mathbb{Z}$-)divisor extending b. We want to modify it (at the special fiber only) so as to arrange $B' \cdot F_i = 0$ for all i. With this in mind, consider the $\mathbb{Q}$-vector space $V = \bigoplus_{i=1}^r \mathbb{Q} \cdot e_i$ endowed with the bilinear pairing determined by

$$e_i \cdot e_j := F_i \cdot F_j.$$

Write $\mathcal{C}_k = \sum_i a_i F_i \in \mathrm{Div}(\mathcal{C})$, and let $w := \sum_i a_i e_i \in V$. By Theorem 2.9, the linear map

$$\begin{array}{rccc} T: & V & \longrightarrow & V^\vee \\ & v & \longmapsto & (-) \cdot v \end{array}$$

has 1-dimensional kernel (spanned by w), and has image contained in

$$U := \{\varphi \in V^\vee : \varphi(w) = 0\}.$$

Thus, $T(V) = U$ by dimension counting. Now, consider the linear functional $\varphi \in V^\vee$ determined by $\varphi(e_i) = B' \cdot F_i$, and note that

$$\varphi(w) = B' \cdot \mathcal{C}_k = \deg(\mathscr{O}_{\mathcal{C}_k}(B'_k)) = \deg(\mathscr{O}_{\mathcal{C}_K}(B'_K)) = \deg(b) = 0,$$

where the nontrivial equality above holds since $\mathscr{O}_{\mathcal{C}}(B)$ is a line bundle on the *flat* R-scheme $\mathcal{C}$. Thus, $\varphi \in U = T(V)$, so there exists some vertical divisor $C \in \mathrm{VDiv}(\mathcal{C}) \otimes \mathbb{Q}$ so that $C \cdot F_i = B' \cdot F_i$ for all i. Thus, $B := B' - C \in \mathrm{Div}(\mathcal{C}) \otimes \mathbb{Q}$ is our desired extension. ∎

Construction 3.3. Let $a \in Z^0(C/K)$ and $b \in \mathrm{Div}^0(C/K)$ be relative prime divisors. Let $A, B \in \mathrm{Div}(\mathcal{C}) \otimes \mathbb{Q}$ be extensions of a, b, respectively, as in Lemma 3.2. Set

$$\langle a, b \rangle_k := -(A \cdot B) \log q,$$

where $q = \#k$.

Remark 3.4. Above, as long as we choose B to have 0 intersection with every vertical divisor, we can compute $\langle a, b\rangle_k$ with any extension A of a (even the naive one). ◦

Warning 3.5. Neither A nor B above is fibral, so Theorem 2.7 does not apply to their intersection product (which is defined via Abuse of Notation 2.8). •

Proposition 3.6. *$\langle -, -\rangle_k$ defined above satisfies properties **(1)** – **(4)** of Theorem 1.6.*

Proof. First note that $\langle -, -\rangle_k$ is well-defined since its value is insensitive to modifying A or B by multiplies of $\mathcal{C}_k$ (as $A \cdot \mathcal{C}_k = 0 = B \cdot \mathcal{C}_k$). Properties **(1)**,**(2)** follow from the corresponding properties of the intersection pairing.

For **(3)**,**(4)**, we perform the following computation. Fix $a \in Z^0(C/K)$ and $b \in \operatorname{Div}^0(C/K)$, and extend these to $A, B \in \operatorname{Div}(\mathcal{C}) \otimes \mathbb{Q}$ s.t. $A \cdot F = 0 = B \cdot F$ for any $F \in \operatorname{VDiv}(\mathcal{C})$. Since A, B share no irreducible components ($\impliedby$ a, b being relatively prime), there is an open $U \subset \mathcal{C}$ containing all the generic points of components of A and a section $s \in \Gamma(U, \mathscr{O}_{\mathcal{C}}(B))$ whose zero scheme is $(s) = U \cap B$. Write $A = \sum a_i \overline{P_i}$ with $P_i \in \mathcal{C}_K$. Note that, since $\mathcal{C}$ is proper, the K-point P_i extends uniquely to an $\mathscr{O}_K$-point $Z_i : \operatorname{Spec} R \to \mathcal{C}$; the image of this extension is precisely $\overline{Z}_i$. Now, set $(P_i)_k := Z_i((\pi)) \in \mathcal{C}_k$, and stare at the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[rr, bend left, "P_i"] & \operatorname{Spec} R \arrow[r, "Z_i"'] & \mathcal{C} \end{tikzcd}$$

until you are convinced that $\operatorname{ord}_{Z_i}(s) = v(s(P_i))$ where $\operatorname{ord}_{Z_i} : K \to \mathbb{Z}$ is the valuation on $\mathscr{O}_{Z_i,(P_i)_k} \simeq R_{(\pi)} = R$. Finally, we compute (recall Remark 2.5)

$$\begin{aligned} -(A \cdot B) \log q &= \sum_i \sum_{x \in Z_i \cap B} -a_i \operatorname{ord}_{Z_i}(s) \log q \\ &= \sum_i \sum_{x \in Z_i \cap \mathcal{C}_k} -a_i \operatorname{ord}_{Z_i}(s) \log q \\ &= \sum_i -a_i v(s(P_i)) \log q \\ &= \log |s(a)| . \end{aligned}$$

Taking $b = \operatorname{div}(f)$, this gives **(3)** immediately. **(4)** follows from the fact that s gives a morphism $C \setminus \operatorname{supp}(b) \to \mathbb{A}^1$ (which induces a continuous map $C(K) \setminus \operatorname{supp}(b) \to K$ on K-points in the analytic topology) and that $\log|\cdot| : K \to \mathbb{R}$ is continuous. ■

4 Relation to Global Heights

I haven't been able to find a proof of Theorem 3 which is understandable and which treats local heights in the same fashion as [Gro86],[3] so I have nothing useful to say here... At best, I can offer up this.

Note 2. There's a "sketch" of a proof of Theorem 3 in Serre's 'Lectures on the Mordell Weil Theorem'. There's allegedly a full proof of it in [Lan83, Chapter 11] and in [BG06, Section 9.5], but I haven't

[3]Arguably, I haven't seen a proof which satisfies either of these requirements

managed to piece out exactly where in these texts the proof is (they take a different approach to local height functions than we did, so it's not so easy for me to tell what's going on in them).

Slightly more seriously, one might hope to prove it by showing that, for a nice curve C/K over a global field K, the function

$$\begin{array}{rccc} f: & \mathrm{Div}^0(C) & \longrightarrow & \mathbb{R} \\ & a & \longmapsto & \displaystyle\sum_v \langle a, a\rangle_v \end{array}$$

satisfies the defining properties of the Néron-Tate height associated to the divisor 2Θ. That is, f is (up to $O(1)$) a Weil height function, and also $f(na) = n^2 f(a)$ for all $a \in \mathrm{Div}^0(C)$ and $n \in \mathbb{Z}$. I guess it's also worth noting that f vanishes on principal divisors (and so descends to a function on $J(K)$) by the product formula.

That $f(na) = n^2 f(a)$ follows from bilinearity of the local symbols. However, I am not sure why f is a Weil height function for 2Θ...

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