GENERAL STRATEGY FOR ALGEBRAIC DEGENERACY

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The goal is to prove Theorem 7 below, which gives a criterion for every entire curve on a compact complex manifold to be degenerate (i.e. to land in some proper submanifold). Its statement is rather hefty, so, at least for my own personal benefit as a newcomer to the area, let's begin by very briefly recalling some of the constructions/results that have appeared in the past few talks.

Setup 1. Let X be an n-dimensional compact complex manifold, also viewed as the directed manifold (X, V) with $V \coloneqq T_X$. ⊚

(feel free to imagine that $V \subsetneq T_X$ if you want)

Recall 2. For $k \geq 1$, we define the `space of k-jets` of X to be the bundle $p_k\colon J_kT_X \to X$ parameterizing k-jets of curves $f\colon \mathbb{C} \to X$.[1] This is a $(\mathbb{C}^n)^k$-fiber bundle over X.[2] ⊙

Recall 3. There are also *projectivized jet bundles.*

- The `projectivized first jet bundle` is (X_1, V_1) where $X_1 = \mathbb{P}(V) \coloneqq \mathbf{Proj}(\operatorname{Sym} V^\vee) \xrightarrow{\pi} X$ is the projective bundle of *lines in* V,[3] equipped with the tautological line subbundle $\mathscr{O}_{X_1}(-1) \subset \pi^*V$, and V_1 is defined via:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X_1/X} & \longrightarrow & V_1 & \longrightarrow & \mathscr{O}_{X_1}(-1) & \longrightarrow & 0 \\ & & \| & & \downarrow & \lrcorner & \downarrow & & \\ 0 & \longrightarrow & T_{X_1/X} & \longrightarrow & T_{X_1} & \longrightarrow & \pi^*T_X & \longrightarrow & 0. \end{array}$$

- Iterating the above leads to the `projectivized k-jet bundle` $\pi_k\colon (X_k, V_k) \to (X, V)$.[4] ⊙

Recall 4. Le $\mathbb{G}_k$ denote the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, i.e. germs of biholomorphic maps

$$t \mapsto a_1 t + a_2 t^2 + \cdots + a_k t^k \ \text{ with } \ a_i \in \mathbb{C}^\times \ \text{ and } \ a_j \in \mathbb{C}, j \geq 2.$$

This $\mathbb{G}_k$ acts on $J_kV = J_kT_X$ via reparameterizing k-jets, and this leads one to defined the `bundle of invariant jet differentials of order k and weighted degree m` $E_{k,m}V^\vee \longrightarrow X$ whose fibers are spaces of complex valued polynomials $Q(f', f'', \dots, f^{(k)})$ on the fibers of J_kV satisfying

$$Q\Big((f \circ \varphi)', (f \circ \varphi)'', \dots, (f \circ \varphi)^{(k)}\Big) = \varphi'(0)^m Q(f', f'', \dots, f^{(k)})$$

for all $\varphi \in \mathbb{G}_k$. Note that sections of $E_{k,m}V^\vee$ may be viewed as ($\mathbb{G}_k$-equivariant) maps $J_kV \to \mathbb{C}$. ⊙

Recall 5 (Duc's talk, [DR16, Theorem 3.3.1])**.**

(1) There is an embedding $J_kV^{\mathrm{reg}}/\mathbb{G}_k \hookrightarrow X_k$ over X which identifies $J_kV^{\mathrm{reg}}/\mathbb{G}_k \xhookrightarrow{\sim} X_k^{\mathrm{reg}}$. Here, a `regular k-jet` is one with non-vanishing first derivative.
(so X_k is a relative compactification of $J_kV^{\mathrm{reg}}/\mathbb{G}_k$)

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[1] Equivalence classes of holomorphic maps $f\colon (\mathbb{C}, 0) \to (X, f(0))$ where $f \sim g$ iff $f^{(j)}(0) = g^{(j)}(0)$ for $0 \leq j \leq k$ in some ($\iff$ all) holomorphic coordinate system on X near $f(0)$.

[2] Vector bundle if $k \leq 1$, in which cases $J_1T_X = T_X$ and $J_0T_X = X$.

[3] Some authors write this space as $\mathbb{P}(\Omega^1) = \mathbf{Proj}(\operatorname{Sym} \Omega^1)$ instead.

[4] E.g. $(X_2, V_2) = ((X_1)_1, (V_1)_1)$ and so on...

(2) The direct image sheaf

$$(\pi_k)_* \mathscr{O}_{X_k}(m) \simeq \mathscr{O}(E_{k,m}V^\vee)$$

is identified with the sheaf of holomorphic sections of $E_{k,m}V^\vee \to X$.
(Compare: $p_*\mathscr{O}_{\mathbb{P}(E)}(m) \simeq \mathrm{Sym}^m(E^\vee)$, where $\mathbb{P}(E) = \mathbf{Proj}(\mathrm{Sym}\, E^\vee) \xrightarrow{p} X$) ⊙

Recall 6 (Jit Wu's talk, [DR16, Corollary 4.2.5])**.** Assume there are integers $k, m \geq 1$ and an ample line bundle A on X such that

$$\mathrm{H}^0\big(X_k, \mathscr{O}_{X_k}(m) \otimes \pi_k^* A^{-1}\big) \simeq \mathrm{H}^0\big(X, E_{k,m}V^\vee \otimes A^{-1}\big)$$

has nonzero sections $\sigma_1, \dots, \sigma_N$ with base local $Z = \bigcap_{i=1}^N \{\sigma_i = 0\}$. Then, every entire curve $f\colon \mathbb{C} \to X$ tangent to V satisfies $f_{[k]}(\mathbb{C}) \subset Z$. ⊙

Slogan (Recall 6 in words)**.** If X has a jet differential σ valued in an anti-ample line bundle (e.g. A^{-1}), then the image of any entire curve satisfies the corresponding differential equation (i.e. $f_{[k]}(\mathbb{C}) \subset \{\sigma = 0\} \subset X_k$).

Theorem 7 ([DR16, Theorem 4.3.1])**.** *Suppose there are two ample line bundles A, B on X and integers $k, m > 0$ such that*

(i) *there is a nonzero section $P \in \mathrm{H}^0(X, E_{k,m}T_X^\vee \otimes A^{-1})$.*[5]

(ii) *the* `twisted tangent space` *$T_{J_kT_X} \otimes p_k^* B$ of the space of k-jets*

$$p_k\colon J_kT_X \longrightarrow X$$

is globally generated over its regular part $J_kT_X^{\mathrm{reg}}$ by its global sections. Moreover, suppose one can choose such generating vector fields to be equivariant w.r.t the action of $\mathbb{G}_k \curvearrowright J_kT_X$.

(iii) *the line bundle $A \otimes B^{-m}$ is ample.*

Then, every holomorphic entire curve $f\colon \mathbb{C} \to X$ has image contained in $Y \coloneqq \{P = 0\} \subsetneq X$.

Proof. Let $f\colon \mathbb{C} \to X$ be an entire curve, with lifting $j_k(f)\colon \mathbb{C} \to J_kT_X$. Note that $j_k(f)(\mathbb{C}) \not\subset J_kT_X^{\mathrm{sing}} \coloneqq J_kX \setminus J_kT_X^{\mathrm{reg}}$; otherwise, $f' = 0$ everywhere so f would be constant. Suppose that $f(\mathbb{C}) \not\subset Y$ and choose some $\zeta_0 \in \mathbb{C}$ so that $x_0 \coloneqq f(\zeta_0) \notin Y$. Note that we may and do choose ζ_0 so that $j_k(f)(\zeta_0) \in J_kT_X^{\mathrm{reg}}$. Indeed, this amounts to showing that the intersection

$$f^{-1}(X \setminus Y) \cap j_k(f)^{-1}(J_kT_X^{\mathrm{reg}}) \tag{1}$$

is nonempty. Note that $f^{-1}(X \setminus Y) \subset \mathbb{C}$ is a dense open; $Y \subset X$ is cut out by finitely many holomorphic functions on X and any such function, when pulled back to $\mathbb{C}$, has only finitely many zeros. Furthermore, we saw above that $j_k(f)^{-1}(J_kT_X^{\mathrm{reg}})$ is a nonempty open so the intersection (1) must be nonempty, i.e. there is some $\zeta_0 \in \mathbb{C}$ such that $f(\zeta_0) \notin Y$ and $j_k(f)(\zeta_0) \in J_kT_X^{\mathrm{reg}}$.
Note that we may view P as a $\mathbb{G}_k$-invariant map

$$P\colon J_kT_X \longrightarrow p_k^* A^{-1}.$$

It follows from Recall 6 that $j_k(f)(\zeta_0) \in \{P_{x_0} = 0\} \subset (J_kT_X)_{x_0} = J_kT_{X,x_0}$ (See Fig. 1). Furthermore, $P_{x_0}\colon J_kT_{X,x_0} \to \mathbb{C}$ is *not* identically zero since $x_0 \notin Y$.
Now, the idea is that, because $P_{x_0} \neq 0$, it must have some derivative *not* vanishing at $j_k(f)(\zeta_0) \eqqcolon z_0 \in J_kT_{X,x_0}$. Using **(ii)**, we'll be able to realize find some global vector fields such that differentiating P with respect to them produces a new jet differential Q which is *non-vanishing* at z_0, contradicting Recall 6.
After choosing local coordinates, we can realize P_{x_0} as a polynomial of weighted degree m in nk variables $X_1', \dots, X_n', X_1'', \dots, X_n'', \dots, X_1^{(k)}, \dots, X_n^{(k)}$ chosen so that $z_0 = j_k(f)(\zeta_0)$ corresponds to $X_1' = \dots = X_n^{(k)} = 0$. Since P_{x_0} has weighted degree m, every monomial appearing in it must have (unweighted) total degree $\leq m$. Consider some such monomial of least degree, say of degree $p \leq m$, which we may write as

$$\lambda Y_1 Y_2 \dots Y_p \quad \text{where} \quad \lambda \in \mathbb{C}^\times \quad \text{and} \quad Y_1, \dots, Y_p \in \{X_1', \dots, X_n^{(k)}\}.$$

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[5]This space is isomorphic to $\mathrm{H}^0(X_k, \mathscr{O}_{X_k}(m) \otimes \pi_k^* A^{-1})$; see Recall 5**(1)**.

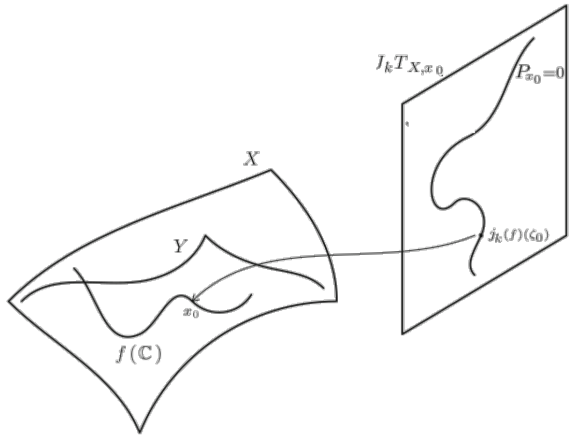

FIGURE 1. The lifting of the curve f, taken (with slight modifications) from [DR16, Fig. 4.1]

Then, $\partial_{Y_p}\dots\partial_{Y_1}P_{x_0}$ is non-vanishing at z_0, so we would like to realize it as the fiber of some global jet differential. Because we arranged to have $z_0 = j_k(f)(\zeta_0) \in J_kT_X^{\mathrm{reg}}$, **(ii)** implies that there are $\mathbb{G}_k$-invariant global vector fields $V_1,\dots,V_p \in \mathrm{H}^0(J_kT_X, T_{J_kT_X} \otimes p_k^*B)$ such that $V_j(z_0) = \partial_{Y_j}|_{z_0}$. Thus, taking Lie derivatives,[6]

$$Q \coloneqq L_{V_p}\dots L_{V_1}P \in \mathrm{H}^0(X, E_{k,m}T_X^\vee \otimes B^p \otimes A^{-1})$$

satisfies $Q(z_0) \neq 0$, contradicting Recall 6 since $A \otimes B^{-p}$ is ample by **(iii)**. ■

Corollary 8. *Use notation as in Theorem 7. Suppose, furthermore, that the effective cone of X is contained in its ample cone (for example, that $\mathrm{Pic}\,X \simeq \mathbb{Z}$). Then, one can choose A such that $\mathrm{codim}_X(Y) \geq 2$.*

Proof. Let D be the divisorial part of $Y = \{P = 0\}$. Then, P can be viewed as a section of $E_{k,m}T_X^\vee \otimes (A \otimes \mathscr{O}_X(D))^{-1}$, and seen as a section of this bundle, it vanishes on no codimension 1 subvariety of X. Since D is ample, $A \otimes \mathscr{O}_X(D)$ is still ample and still satisfies property **(iii)** (w.r.t the same B), so one can apply Theorem 7 to $P \in \mathrm{H}^0(X, E_{k,m}T_X^\vee \otimes (A \otimes \mathscr{O}_X(D))^{-1})$. ■

Corollary 9. *A compact complex surface satisfying the hypotheses of Corollary 8 is Kobayashi hyperbolic.*

(Recall that for *compact* complex manifolds, Kobayashi hyperbolicity is equivalent to the nonexistence of entire curves, see [DR16, Proposition 1.2.1 and Theorem 1.2.2])

Corollary 10. *Let X be a compact complex threefold satisfying the hypotheses of Corollary 8 which does not contain any rational or elliptic curve. Then, X is Kobayashi hyperbolic.*

Proof. Let $f\colon \mathbb{C} \to X$ be an entire curve in X. Then, $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ is an algebraic curve of X admitting a non-constant holomorphic image of $\mathbb{C}$, so it must be rational or elliptic, a contradiction. ■

REFERENCES

[DR16] Simone Diverio and Erwan Rousseau. *Hyperbolicity of projective hypersurfaces*, volume 5 of *IMPA Monographs*. Springer, [Cham], second edition, 2016. 1, 2, 3

[6] I think what's happening here is that $T_{J_kT_X} \otimes p_k^*B \simeq \mathrm{Hom}(\Omega^1_{J_kT_X}, p_k^*B) \simeq \mathrm{Der}(\mathscr{O}_{J_kT_X}, p_k^*B)$ (sheaf of p_k^*B-valued derivations), so given a global section V, we can think of it as a derivation $\mathscr{O}_{J_kT_X} \to p_k^*B$. Such a thing naturally gives rise to a derivation $p_k^*A^{-1} \to p_k^*B \otimes p_k^*A^{-1} = p_k^*(B \otimes A^{-1})$ and L_VP is simply notation for the image of P under this latter derivation. A propri, this is simply a map $L_VP\colon J_kT_X \to p_k^*(B \otimes A^{-1})$, but I think one can check after the fact that it really arises from a k-jet differential of degree m values in $B \otimes A^{-1}$, i.e. from a section of $E_{k,m}T_X^\vee \otimes B \otimes A^{-1}$.