

Problems & Solutions

1. *Migdal-Kadanoff method:* Consider Potts spins $s_i = (1, 2, \dots, q)$, on sites i of a hypercubic lattice, interacting with their nearest neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta_{s_i, s_j} .$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Identify all fixed points and note their stability.

• In $d = 1$, if we average over the q possible values of s_1 , we obtain

$$\sum_{s_1=1}^q e^{K(\delta_{\sigma_1 s_1} + \delta_{s_1 \sigma_2})} = \begin{cases} q - 1 + e^{2K} & \text{if } \sigma_1 = \sigma_2 \\ q - 2 + 2e^K & \text{if } \sigma_1 \neq \sigma_2 \end{cases} = e^{g' + K' \delta_{\sigma_1 \sigma_2}},$$

from which we arrive at the exact recursion relations:

$$e^{K'} = \frac{q - 1 + e^{2K}}{q - 2 + 2e^K}, \quad e^{g'} = q - 2 + 2e^K.$$

To find the fixed points we set $K' = K = K^*$. As in the previous problem, let us introduce the variable $x = e^{K^*}$. Hence, we have to solve the equation

$$x = \frac{q - 1 + x^2}{q - 2 + 2x}, \quad \text{or} \quad x^2 + (q - 2)x - (q - 1) = 0,$$

whose only meaningful solution is $x = 1$, resulting in $K^* = 0$. To check its stability, we consider $K \ll 1$, so that

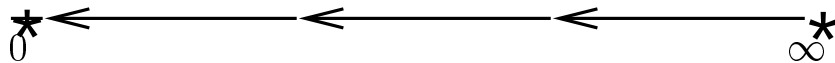
$$K' \simeq \ln \left(\frac{q + 2K + 2K^2}{q + 2K + K^2} \right) \simeq \frac{K^2}{q} \ll K,$$

which indicates that this fixed point is stable.

In addition, $K^* \rightarrow \infty$ is also a fixed point. If we consider $K \gg 1$,

$$e^{K'} \simeq \frac{1}{2} e^K, \quad \implies \quad K' = K - \ln 2 < K,$$

which implies that this fixed point is unstable.



(b) Write down the recursion relation $K'(K)$ in d -dimensions for $b = 2$, using the Migdal-Kadanoff bond moving scheme.

- In the Migdal-Kadanoff approximation, moving bonds strengthens the remaining bonds by a factor 2^{d-1} . Therefore, in the decimated lattice we have

$$e^{K'} = \frac{q - 1 + e^{2 \times 2^{d-1} K}}{q - 2 + 2e^{2^{d-1} K}}.$$

(c) By considering the stability of the fixed points at zero and infinite coupling, prove the existence of a non-trivial fixed point at finite K^* for $d > 1$.

- In the vicinity of the fixed point $K^* = 0$, i.e. for $K \ll 1$,

$$K' \simeq \frac{2^{2d-2} K^2}{q} \ll K,$$

and consequently, this point is again stable. However, for $K^* \rightarrow \infty$, we have

$$e^{K'} \simeq \frac{1}{2} \exp[(2^d - 2^{d-1}) K], \implies K' = 2^{d-1} K - \ln 2 \gg K,$$

which implies that this fixed point is now stable provided that $d > 1$.

- As a result, there must be a finite K^* fixed point, which separates the flows to the other fixed points.



(d) For $d = 2$, obtain K^* and y_t , for $q = 3, 1$, and 0 .

- Let us now discuss a few particular cases in $d = 2$. For instance, if we consider $q = 3$, the non-trivial fixed point is a solution of the equation

$$x = \frac{2 + x^4}{1 + 2x^2}, \quad \text{or} \quad x^4 - 2x^3 - x + 2 = (x - 2)(x^3 - 1) = 0,$$

which clearly yields a non-trivial fixed point at $K^* = \ln 2 \simeq 0.69$. The thermal exponent for this point

$$\left. \frac{\partial K'}{\partial K} \right|_{K^*} = 2^{y_t} = 2 \left[\frac{e^{4K^*}}{e^{4K^*} + 2} - \frac{e^{2K^*}}{1 + 2e^{2K^*}} \right] = \frac{16}{9}, \implies y_t \simeq 0.83,$$

which can be compared to the exact values, $K^* = 1.005$, and $y_t = 1.2$.

By analytic continuation for $q \rightarrow 1$, we obtain

$$e^{K'} = \frac{e^{4K}}{-1 + 2e^{2K}}.$$

The non-trivial fixed point is a solution of the equation

$$x = \frac{x^4}{-1 + 2x^2}, \quad \text{or} \quad (x^3 - 2x^2 + 1) = (x - 1)(x^2 - x - 1) = 0,$$

whose only non-trivial solution is $x = (1 + \sqrt{5})/2 = 1.62$, resulting in $K^* = 0.48$. The thermal exponent for this point

$$\left. \frac{\partial K'}{\partial K} \right|_{K^*} = 2^{y_t} = 4 \left[1 - e^{-K^*} \right], \quad \implies \quad y_t \simeq 0.61.$$

As discussed in the next problem set, the Potts model for $q \rightarrow 1$ can be mapped onto the problem of *bond percolation*, which despite being a purely geometrical phenomenon, shows many features completely analogous to those of a continuous thermal phase transition.

And finally for $q \rightarrow 0$, relevant to *lattice animals* (see PS#9), we obtain

$$e^{K'} = \frac{-1 + e^{4K}}{-2 + 2e^{2K}},$$

for which we have to solve the equation

$$x = \frac{-1 + x^4}{-2 + 2x^2}, \quad \text{or} \quad x^4 - 2x^3 + 2x - 1 = (x - 1)^3(x + 1) = 0,$$

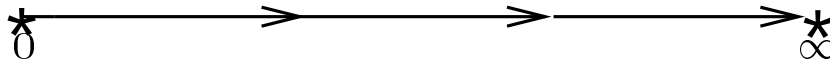
whose only finite solution is the trivial one, $x = 1$. For $q \rightarrow 0$, if $K \ll 1$, we obtain

$$K' \simeq K + \frac{K^2}{2} > K,$$

indicating that this fixed point is now unstable. Note that the first correction only indicates marginal stability ($y_t = 0$). Nevertheless, for $K^* \rightarrow \infty$, we have

$$e^{K'} \simeq \frac{1}{2} \exp[2K], \quad \implies \quad K' = 2K - \ln 2 \gg K,$$

which implies that this fixed point is stable.



2. The Potts model: The *transfer matrix* procedure can be extended to Potts model, where the spin s_i on each site takes q values $s_i = (1, 2, \dots, q)$; and the Hamiltonian is $-\beta\mathcal{H} = K \sum_{i=1}^N \delta_{s_i, s_{i+1}} + K \delta_{s_N, s_1}$.

(a) Write down the transfer matrix and diagonalize it. Note that you do not have to solve a q^{th} order secular equation as it is easy to guess the eigenvectors from the symmetry of the matrix.

- The partition function is

$$Z = \sum_{\{s_i\}} \langle s_1 | T | s_2 \rangle \langle s_2 | T | s_3 \rangle \cdots \langle s_{N-1} | T | s_N \rangle \langle s_N | T | s_1 \rangle = \text{tr}(T^N),$$

where $\langle s_i | T | s_j \rangle = \exp(K\delta_{s_i, s_j})$ is a $q \times q$ transfer matrix. The diagonal elements of the matrix are e^K , while the off-diagonal elements are unity. The eigenvectors of the matrix are easily found by inspection. There is one eigenvectors with all elements equal; the corresponding eigenvalue is $\lambda_1 = e^K + q - 1$. There are also $(q - 1)$ eigenvectors orthogonal to the first, i.e. the sum of whose elements is zero. This corresponding eigenvalues are degenerate and equal to $e^K - 1$. Thus

$$Z = \sum_{\alpha} \lambda_{\alpha}^N = (e^K + q - 1)^N + (q - 1)(e^K - 1)^N.$$

(b) Calculate the free energy per site.

- Since the largest eigenvalue dominates for $N \gg 1$,

$$\frac{\ln Z}{N} = \ln(e^K + q - 1).$$

(c) Give the expression for the correlation length ξ (you don't need to provide a detailed derivation), and discuss its behavior as $T = 1/K \rightarrow 0$.

- Correlations decay as the ratio of the eigenvalues to the power of the separation. Hence the correlation length is

$$\xi = \left[\ln \left(\frac{\lambda_1}{\lambda_2} \right) \right]^{-1} = \left[\ln \left(\frac{e^K + q - 1}{e^K - 1} \right) \right]^{-1}.$$

In the limit of $K \rightarrow \infty$, expanding the above result gives

$$\xi \simeq \frac{e^K}{q} = \frac{1}{q} \exp \left(\frac{1}{T} \right).$$

3. Continuous spins: In the standard $\mathcal{O}(n)$ model, n component unit vectors are placed on the sites of a lattice. The nearest neighbor spins are then connected by a bond $J\vec{s}_i \cdot \vec{s}_j$. In fact, if we are only interested in universal properties, any generalized interaction $f(\vec{s}_i \cdot \vec{s}_j)$ leads to the same critical behavior. By analogy with the Ising model, a suitable choice is

$$\exp[f(\vec{s}_i \cdot \vec{s}_j)] = 1 + (nt)\vec{s}_i \cdot \vec{s}_j,$$

resulting in the so called *loop model*.

(a) Construct a high temperature expansion of the loop model (for the partition function Z) in the parameter t , on a two-dimensional *hexagonal* (honeycomb) lattice.

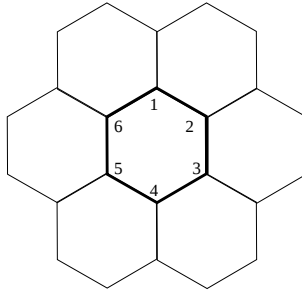
- The partition function for the loop model has the form

$$Z = \int \{\mathcal{D}\mathbf{s}_i\} \prod_{\langle ij \rangle} [1 + (nt)\mathbf{s}_i \cdot \mathbf{s}_j],$$

that we can expand in powers of the parameter t . If the total number of nearest neighbor bonds on the lattice is N_B , the above product generates 2^{N_B} possible terms. Each term may be represented by a graph on the lattice, in which a bond joining spins i and j is included if the factor $\mathbf{s}_i \cdot \mathbf{s}_j$ appears in the term considered. Moreover, each included bond carries a factor of nt . As in the Ising model, the integral over the variables $\{\mathbf{s}_i\}$ leaves only graphs with an even number of bonds emanating from each site, because

$$\int d\mathbf{s} s_\alpha = \int d\mathbf{s} s_\alpha s_\beta s_\gamma = \cdots = 0.$$

In a honeycomb lattice, as plotted below, there are only 1, 2, or 3 bonds emerging from each site. Thus the only contributing graphs are those with two bonds at each site, which, as any bond can only appear once, are closed *self-avoiding loops*.



While the honeycomb lattice has the advantage of not allowing intersections of loops at a site, the universal results are equally applicable to other lattices.

We shall rescale all integrals over spin by the n -dimensional solid angle, such that $\int d\mathbf{s} = 1$. Since $s_\alpha s_\alpha = 1$, it immediately follows that

$$\int d\mathbf{s} s_\alpha s_\beta = \frac{\delta_{\alpha\beta}}{n},$$

resulting in

$$\int d\mathbf{s}' (s_\alpha s'_\alpha)(s'_\beta s''_\beta) = \frac{1}{n} s_\alpha s''_\alpha.$$

A sequence of such integrals forces the components of the spins around any loop to be the same, and there is a factor n when integrating over the last spin in the loop, for instance

$$\int \{\mathcal{D}\mathbf{s}_i\} (s_{1\alpha} s_{2\alpha})(s_{2\beta} s_{3\beta})(s_{3\gamma} s_{4\gamma})(s_{4\delta} s_{5\delta})(s_{5\eta} s_{6\eta})(s_{6\nu} s_{1\nu}) = \frac{\delta_{\alpha\beta}\delta_{\beta\gamma}\delta_{\gamma\delta}\delta_{\delta\eta}\delta_{\eta\nu}\delta_{\alpha\nu}}{n^6} = \frac{n}{n^6}.$$

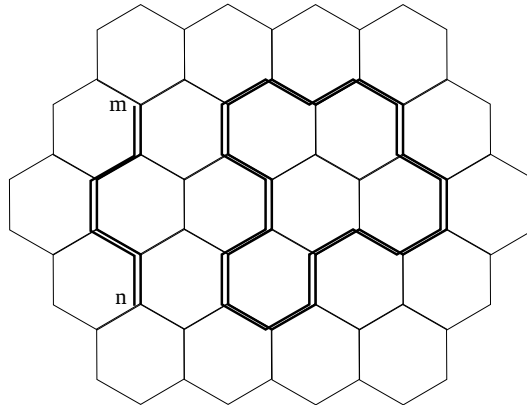
Since each bond carries a factor of nt , each loop finally contributes a factor $n \times t^\ell$, where ℓ is the number of bonds in the loop. The partition function may then be written as

$$Z = \sum_{\text{self-avoiding loops}} n^{N_\ell} t^{N_b},$$

where the sum runs over distinct disconnected or self-avoiding loops collections with a bond fugacity t , and N_ℓ , N_b are the number of loops, and the number of bonds in the graph, respectively. Note that, as we are only interested in the critical behavior of the model, any global analytic prefactor is unimportant.

(b) Show that the limit $n \rightarrow 0$ describes the configurations of a single self-avoiding polymer on the lattice.

- While $Z = 1$, at exactly $n = 0$, one may obtain non-trivial information by considering the limit $n \rightarrow 0$. The leading term ($\mathcal{O}(n^1)$) when $n \rightarrow 0$ picks out just those configurations with a single self-avoiding loop, i.e. $N_\ell = 1$.



The correlation function can also be calculated graphically from

$$G_{\alpha\beta}(n-m) = \langle s_{n\alpha} s_{m\beta} \rangle = \frac{1}{Z} \int \{ \mathcal{D} \mathbf{s}_i \} s_{n\alpha} s_{m\beta} \prod_{\langle ij \rangle} [1 + (nt) \mathbf{s}_i \cdot \mathbf{s}_j].$$

After disregarding any global prefactor, and taking the limit $n \rightarrow 0$, the only surviving graph consists of a single line going from n to m , and the index of all the spins along the line is fixed to be the same. All other possible graphs disappear in the limit $n \rightarrow 0$. Therefore, we are left with a sum over self-avoiding walks that go from n to m , each carrying a factor t^ℓ , where ℓ indicates the length of the walk. If we denote by $W_\ell(R)$ the number of self-avoiding walks of length ℓ whose end-to-end distance is R , we can write that

$$\sum_{\ell} W_\ell(R) t^\ell = \lim_{n \rightarrow 0} G(R).$$

As in the case of phantom random walks, we expect that for small t , small paths dominate the behavior of the correlation function. As t increases, larger paths dominate the sum, and, ultimately, we will find a singularity at a particular t_c , at which arbitrarily long paths become possible.

Although we presented the mapping of self-avoiding walks to the $n \rightarrow 0$ limit of the $\mathcal{O}(n)$ model for a honeycomb lattice, the critical behavior should be universal, and therefore independent of this lattice choice. What is more, various scaling properties of self-avoiding walks can be deduced from the $\mathcal{O}(n)$ model with $n \rightarrow 0$. Let us, for instance, characterize the mean square end-to-end distance of a self-avoiding walk, defined as

$$\langle R^2 \rangle = \frac{1}{W_\ell} \sum_R R^2 W_\ell(R),$$

where $W_\ell = \sum_R W_\ell(R)$ is the total number of self-avoiding walks of length ℓ .

The singular part of the correlation function decays with separation R as $G \propto |R|^{-(d-2+\eta)}$, up to the correlation length ξ , which diverges as $\xi \propto (t_c - t)^{-\nu}$. Hence,

$$\sum_R R^2 G(R) \propto \xi^{d+2-(d-2+\eta)} = (t_c - t)^{-\nu(4-\eta)} = (t_c - t)^{-\gamma-2\nu}.$$

We noted above that $G(t, R)$ is the generating function of $W_\ell(R)$, in the sense that $\sum_\ell W_\ell(R) t^\ell = G(t, R)$. Similarly $\sum_\ell W_\ell t^\ell$ is the generating function of W_ℓ , and is related to the susceptibility χ , by

$$\sum_\ell W_\ell t^\ell = \sum_R G(R) = \chi \propto (t_c - t)^{-\gamma}.$$

To obtain the singular behavior of W_ℓ from its generating function, we perform a Taylor expansion of $(t_c - t)^{-\gamma}$, as

$$\sum_{\ell} W_{\ell} t^{\ell} = t_c^{-\gamma} \left(1 - \frac{t}{t_c}\right)^{-\gamma} = t_c^{-\gamma} \sum_{\ell} \frac{\Gamma(1-\gamma)}{\Gamma(1+\ell)\Gamma(1-\gamma-\ell)} \left(\frac{t}{t_c}\right)^{\ell},$$

which results in

$$W_{\ell} = \frac{\Gamma(1-\gamma)}{\Gamma(1+\ell)\Gamma(1-\gamma-\ell)} t_c^{-\ell-\gamma}.$$

After using that $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$, considering $\ell \rightarrow \infty$, and the asymptotic expression of the gamma function, we obtain

$$W_{\ell} \propto \frac{\Gamma(\gamma+\ell)}{\Gamma(1+\ell)} t_c^{-\ell} \propto \ell^{\gamma-1} t_c^{-\ell},$$

and, similarly one can estimate $\sum_R R^2 W_{\ell}(R)$ from $\sum_R R^2 G(R)$, yielding

$$\langle R^2 \rangle \propto \frac{\ell^{2\nu+\gamma-1} t_c^{-\ell}}{\ell^{\gamma-1} t_c^{-\ell}} = \ell^{2\nu}.$$

Setting $n = 0$ in the results of the ϵ -expansion for the $\mathcal{O}(n)$ model, for instance, gives the exponent $\nu = 1/2 + \epsilon/16 + \mathcal{O}(\epsilon^2)$, characterizing the mean square end-to-end distance of a self-avoiding polymer as a function of its length ℓ , rather than $\nu_0 = 1/2$ which describes the scaling of phantom random walks. Because of self-avoidance, the (polymeric) walk is swollen, giving a larger exponent ν . The results of the first order expansion for $\epsilon = 1, 2$, and 3 , in $d = 3, 2$, and 1 are $0.56, 0.625$, and 0.69 , to be compared to $0.59, 3/4$ (exact), and 1 (exact).

4. Potts model I: Consider Potts spins $s_i = (1, 2, \dots, q)$, interacting via the Hamiltonian $-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta_{s_i, s_j}$.

(a) To treat this problem graphically at high temperatures, the Boltzmann weight for each bond is written as

$$\exp(K\delta_{s_i, s_j}) = C(K) [1 + T(K)g(s_i, s_j)],$$

with $g(s, s') = q\delta_{s, s'} - 1$. Find $C(K)$ and $T(K)$.

• To determine the two unknowns $C(K)$ and $T(K)$, we can use the expressions

$$\begin{cases} e^K = C [1 + T(q-1)] & \text{if } s_i = s_j \\ 1 = C [1 - T] & \text{if } s_i \neq s_j \end{cases},$$

from which we obtain

$$T(K) = \frac{e^K - 1}{e^K + q - 1}, \quad \text{and} \quad C(K) = \frac{e^K + q - 1}{q}.$$

(b) Show that

$$\sum_{s=1}^q g(s, s') = 0, \quad \sum_{s=1}^q g(s_1, s)g(s, s_2) = qg(s_1, s_2), \quad \text{and} \quad \sum_{s, s'=1}^q g(s, s')g(s', s) = q^2(q-1).$$

• Moreover, it is easy to check that

$$\begin{aligned} \sum_{s=1}^q g(s, s') &= q - 1 - (q - 1) = 0, \\ \sum_{s=1}^q g(s_1, s)g(s, s_2) &= \sum_{s=1}^q [q^2\delta_{s_1 s}\delta_{s_2 s} - q(\delta_{s_1 s} + \delta_{s_2 s}) + 1] = q(q\delta_{s_1 s_2} - 1) = qg(s_1, s_2), \\ \sum_{s, s'=1}^q g(s, s')g(s', s) &= \sum_{s, s'=1}^q [q^2\delta_{ss'}\delta_{ss'} - 2q\delta_{ss'} + 1] = q^3 - 2q^2 + q^2 = q^2(q-1). \end{aligned}$$

(c) Use the above results to calculate the free energy, and the correlation function $\langle g(s_m, s_n) \rangle$ for a one-dimensional chain.

• The factor $T(K)$ will be our high temperature expansion parameter. Each bond contributes a factor $Tg(s_i, s_j)$ and, since $\sum_s g(s, s') = 0$, there can not be only one bond per any site. As in the Ising case considered in lectures, each bond can only be considered once, and the only graphs that survive have no dangling bonds. As a result, for a one-dimensional chain, with for instance open boundary conditions, it is impossible to draw any acceptable graph, and we obtain

$$Z = \sum_{\{s_i\}} \prod_{\langle ij \rangle} C(K) [1 + T(K)g(s_i, s_j)] = C(K)^{N-1} q^N = q(e^K + q - 1)^{N-1}.$$

Ignoring the boundary effects, i.e., that there are $N - 1$ bonds in the chain, the free energy per site is obtained as

$$-\frac{\beta F}{N} = \ln(e^K + q - 1).$$

With the same method, we can also calculate the correlation function $\langle g(s_n s_m) \rangle$. To get a nonzero contribution, we have to consider a graph that directly connects these two sites. Assuming that $n > m$, this gives

$$\begin{aligned} \langle g(s_n s_m) \rangle &= \frac{C(K)^N}{Z} \sum_{\{s_i\}} g(s_n s_m) \prod_{\langle ij \rangle} [1 + T(K)g(s_i, s_j)] \\ &= \frac{C(K)^N}{Z} T(K)^{n-m} \sum_{\{s_i\}} g(s_n s_m) g(s_m, s_{m+1}) \cdots g(s_{n-1}, s_n) \\ &= \frac{C(K)^N}{Z} T(K)^{n-m} q^{n-m+1} (q-1) q^{N-(n-m)-1} = T^{n-m} (q-1) \end{aligned}$$

where we have used the relationships obtained in (b).

(d) Calculate the partition function on the square lattice to order of T^4 . Also calculate the first term in the low-temperature expansion of this problem.

- The first term in the high temperature series for a square lattice comes from a square of 4 bonds. There are a total of N such squares. Therefore,

$$Z = \sum_{\{s_i\}} \prod_{\langle ij \rangle} C(K) [1 + T(K)g(s_i, s_j)] = C(K)^{2N} q^N [1 + NT(K)^4(q-1) + \dots].$$

Note that any closed loop involving ℓ bonds without intersections contributes $T^\ell q^\ell (q-1)$.

On the other hand, at low temperatures, the energy is minimized by the spins all being in one of the q possible states. The lowest energy excitation is a single spin in a different state, resulting in an energy cost of $K \times 4$ with a degeneracy factor $N \times (q-1)$, resulting in

$$Z = qe^{2NK} [1 + N(q-1)e^{-4K} + \dots].$$

(e) By comparing the first terms in low- and high-temperature series, find a duality rule for Potts models. Don't worry about higher order graphs, they will work out! Assuming a single transition temperature, find the value of $K_c(q)$.

- Comparing these expansions, we find the following duality condition for the Potts model

$$e^{-\tilde{K}} = T(K) = \frac{e^K - 1}{e^K + q - 1}.$$

This duality rule maps the low temperature expansion to a high temperature series, or vice versa. It also maps pairs of points, $\tilde{K} \Leftrightarrow K$, since we can rewrite the above relationship in a symmetric way

$$(e^{\tilde{K}} - 1)(e^K - 1) = q,$$

and consequently, if there is a single singular point K_c , it must be self-dual point,

$$K_c = \tilde{K}_c, \quad \implies \quad K_c = \ln(\sqrt{q} + 1).$$

(f) How do the higher order terms in the high-temperature series for the Potts model differ from those of the Ising model? What is the fundamental difference that sets apart the graphs for $q = 2$? (This is ultimately the reason why only the Ising model is solvable.)

- As mentioned in lectures, the Potts model with $q = 2$ can be mapped to the Ising model by noticing that $\delta_{ss'} = (1 + ss')/2$. However, higher order terms in the high-temperature series of the Potts model involve, in general, graphs with *three* or more bonds emanating from each site. These configurations do not correspond to a random walk, not even a

constrained one as introduced in class for the 2d-Ising model on a square lattice. The quantity

$$\sum_{s_1=1}^q g(s_1, s_2)g(s_1, s_3)g(s_1, s_4) = q^3\delta_{s_2 s_3}\delta_{s_2 s_4} - q^2(\delta_{s_2 s_3} + \delta_{s_2 s_4} + \delta_{s_3 s_4}) + 2q,$$

is always zero when $q = 2$ (as can be easily checked for any possible state of the spins s_2, s_3 and s_4), but is in general different from zero for $q > 2$. This is the fundamental difference that ultimately sets apart the case $q = 2$. Note that the corresponding diagrams in the low temperature expansion involve adjacent regions in 3 (or more) distinct states.

5. Potts model II: An alternative expansion is obtained by starting with

$$\exp[K\delta(s_i, s_j)] = 1 + v(K)\delta(s_i, s_j),$$

where $v(K) = e^K - 1$. In this case, the sum over spins *does not* remove any graphs, and all choices of distributing bonds at random on the lattice are acceptable.

(a) Including a magnetic field $h \sum_i \delta_{s_i, 1}$, show that the partition function takes the form

$$Z(q, K, h) = \sum_{\text{all graphs}} \prod_{\text{clusters } c \text{ in graph}} \left[v^{n_b^c} \times \left(q - 1 + e^{hn_s^c} \right) \right],$$

where n_b^c and n_s^c are the numbers of bonds and sites in cluster c . This is known as the *random cluster expansion*.

• Including a symmetry breaking field along direction 1, the partition function

$$Z = \sum_{\{s_i\}} \prod_{\langle ij \rangle} [1 + v(K)\delta(s_i, s_j)] \prod_i e^{h\delta_{s_i, 1}},$$

can be expanded in powers of $v(K)$ as follows. As usual, if there is a total number N_B of nearest neighbor bonds on the lattice, the product over bonds generates 2^{N_B} possible terms. Each term may be represented by a graph on the lattice, in which a bond joining sites i and j is included if the factor $v\delta(s_i, s_j)$ appears in the term considered. Each included bond carries a factor $v(K)$, as well as a delta function enforcing the equality of the spins on the sites which it connects. In general, these bonds form clusters of different sizes and shapes, and within each cluster, the delta functions force the spins at each vertex to be the same. The sum $\sum_{\{s_i\}}$ therefore gives a factor of $(q - 1) + e^{hn_s^c}$ for each cluster c , where n_s^c is the number of point in the cluster. The partition function may then be written as

$$Z(q, v, h) = \sum_{\text{all graphs}} \prod_{\text{clusters in graph}} \left[v(K)^{n_b^c} \left(q - 1 + e^{hn_s^c} \right) \right],$$

point, which is a continuous geometrical phase transition. The analog of the critical temperature is played by the percolation threshold p_c , which we can calculate using duality as $p_c = 1/2$ (after noting that $v^* = 1$).

An alternative way of obtaining this threshold is to find a duality rule for the percolation problem itself: One can similarly think of the problem in terms of empty bonds with a corresponding probability q . As p plays the role of temperature, there is a mapping of low p to high q or vice versa, and such that $q = 1 - p$. The self-dual point is then obtained by setting $p^* = 1 - p^*$, resulting in $p^* = 1/2$.

(c) Show that in the limit $q \rightarrow 0$, only a single connected cluster contributes to leading order. The enumeration of all such clusters is known as listing *branched lattice animals*.

- The partition function $Z(q, v, h)$ goes to zero at $q = 0$, but again information about geometrical lattice structure can be obtained by taking the limit $q \rightarrow 0$ in an appropriate fashion. In particular, if we set $v = q^a x$, then

$$Z(q, v = xq^a, h = 0) = \sum_{\text{all graphs}} x^{N_b} q^{N_c + aN_b},$$

where N_b and N_c are the total number of bonds and clusters. The leading dependence on q as $q \rightarrow 0$ comes from graphs with the lowest number of $N_c + aN_b$, and depends on the value of a . For $0 < a < 1$, these are the *spanning trees*, which connect all sites of the lattice (hence $N_c = 1$) and that enclose no loops (hence $N_b = N - 1$). Such spanning trees have a power of $x^{a(N-1)} q^{aN-a+1}$, and all other graphs have higher powers of q . For $a = 0$ one can add bonds to the spanning cluster (creating loops) without changing the power, as long as all sites remain connected in a single cluster. These have a relation to a problem referred to as *branched lattice animals*.

6. Potts duality: Consider Potts spins, $s_i = (1, 2, \dots, q)$, placed on the sites of a *square lattice* of N sites, interacting with their nearest-neighbors through a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta_{s_i, s_j}.$$

(a) By comparing the first terms of high and low temperature series, or by any other method, show that the partition function has the property

$$Z(K) = qe^{2NK} \Xi[e^{-K}] = q^{-N} [e^K + q - 1]^{2N} \Xi\left[\frac{e^K - 1}{e^K + (q - 1)}\right],$$

for some function Ξ , and hence locate the critical point $K_c(q)$.

- The low temperature series takes the form

$$Z = qe^{2NK} [1 + N(q-1)e^{-4K} + \dots] \equiv qe^{2NK} \Xi [e^{-K}],$$

while at high temperatures

$$\begin{aligned} Z &= \left[\frac{e^K + q - 1}{q} \right]^{2N} q^N \left[1 + N(q-1) \left(\frac{e^K - 1}{e^K + q - 1} \right)^4 + \dots \right] \\ &\equiv q^{-N} [e^K + q - 1]^{2N} \Xi \left[\frac{e^K - 1}{e^K + q - 1} \right]. \end{aligned}$$

Both of the above series for Ξ are in fact the same, leading to the duality condition

$$e^{-\tilde{K}} = \frac{e^K - 1}{e^K + q - 1},$$

and a critical (self-dual) point of

$$K_c = \tilde{K}_c, \quad \implies \quad K_c = \ln(\sqrt{q} + 1).$$

(b) Starting from the duality expression for $Z(K)$, derive a similar relation for the internal energy $U(K) = \langle \beta \mathcal{H} \rangle = -\partial \ln Z / \partial \ln K$. Use this to calculate the exact value of U at the critical point.

- The duality relation for the partition function gives

$$\ln Z(K) = \ln q + 2NK + \ln \Xi [e^{-K}] = -N \ln q + 2N \ln [e^K + q - 1] + \ln \Xi \left[\frac{e^K - 1}{e^K + q - 1} \right].$$

The internal energy $U(K)$ is then obtained from

$$\begin{aligned} -\frac{U(K)}{K} &= \frac{\partial}{\partial K} \ln Z(K) = 2N - e^{-K} \ln \Xi' [e^{-K}] \\ &= 2N \frac{e^K}{e^K + q - 1} + \frac{qe^K}{(e^K + q - 1)^2} \ln \Xi' \left[\frac{e^K - 1}{e^K + q - 1} \right]. \end{aligned}$$

$\ln \Xi'$ is the derivative of $\ln \Xi$ with respect to its argument, whose value is not known in general. However, at the critical point K_c , the arguments of $\ln \Xi'$ from the high and low temperature forms of the above expression are the same. Substituting $e^{K_c} = 1 + \sqrt{q}$, we obtain

$$2N - \frac{\ln \Xi'_c}{1 + \sqrt{q}} = \frac{2N}{\sqrt{q}} + \frac{\ln \Xi'_c}{1 + \sqrt{q}}, \quad \implies \quad \ln \Xi'_c = \frac{q-1}{\sqrt{q}} N,$$

and,

$$-\frac{U(K_c)}{K_c} = N \left(2 - \frac{q-1}{\sqrt{q}+q} \right), \implies U(K_c) = NK_c \frac{\sqrt{q}+1}{\sqrt{q}}.$$

7. Anisotropic Random Walks: Consider the ensemble of all random walks on a square lattice starting at the origin (0,0). Each walk has a weight of $t_x^{\ell_x} \times t_y^{\ell_y}$, where ℓ_x and ℓ_y are the number of steps taken along the x and y directions respectively.

(a) Calculate the total weight $W(x, y)$, of all walks terminating at (x, y) . Show that W is well defined only for $\bar{t} = (t_x + t_y)/2 < t_c = 1/4$.

• Defining $\langle 0, 0 | W(\ell) | x, y \rangle$ to be the weight of all walks of ℓ steps terminating at (x, y) , we can follow the steps in sec.VI.F of the lecture notes. In the anisotropic case, Eq.(VI.47) (applied ℓ times) is trivially recast into

$$\begin{aligned} \langle x, y | T^\ell | q_x, q_y \rangle &= \sum_{x', y'} \langle x, y | T^\ell | x', y' \rangle \langle x', y' | q_x, q_y \rangle \\ &= (2t_x \cos q_x + 2t_y \cos q_y)^\ell \langle x, y | q_x, q_y \rangle, \end{aligned}$$

where $\langle x, y | q_x, q_y \rangle = e^{iq_x x + iq_y y} / \sqrt{N}$. Since $W(x, y) = \sum_\ell \langle 0, 0 | W(\ell) | x, y \rangle$, its Fourier transform is calculated as

$$\begin{aligned} W(q_x, q_y) &= \sum_\ell \sum_{x, y} \langle 0, 0 | T^\ell | x, y \rangle \langle x, y | q_x, q_y \rangle \\ &= \sum_\ell (2t_x \cos q_x + 2t_y \cos q_y)^\ell = \frac{1}{1 - (2t_x \cos q_x + 2t_y \cos q_y)}. \end{aligned}$$

Finally, Fourier transforming back gives

$$W(x, y) = \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} W(q_x, q_y) e^{-iq_x x - iq_y y} = \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2} \frac{e^{-iq_x x - iq_y y}}{1 - (2t_x \cos q_x + 2t_y \cos q_y)}.$$

Note that the summation of the series is legitimate (for all q 's) only for $2t_x + 2t_y < 1$, i.e. for $\bar{t} = (t_x + t_y)/2 < t_c = 1/4$.

(b) What is the shape of a curve $W(x, y) = \text{constant}$, for large x and y , and close to the transition?

• For x and y large, the main contributions to the above integral come from small q 's. To second order in q_x and q_y , the denominator of the integrand reads

$$1 - 2(t_x + t_y) + t_x q_x^2 + t_y q_y^2.$$

Then, with $q'_i \equiv \sqrt{t_i} q_i$, we have

$$W(x, y) \approx \int_{-\infty}^{\infty} \frac{d^2 q'}{(2\pi)^2 \sqrt{t_x t_y}} \frac{e^{-i \mathbf{q}' \cdot \mathbf{v}}}{1 - 2(t_x + t_y) + \mathbf{q}'^2},$$

where we have extended the limits of integration to infinity, and $\mathbf{v} = \left(\frac{x}{\sqrt{t_x}}, \frac{y}{\sqrt{t_y}} \right)$. As the denominator is rotationally invariant, the integral depends only on the magnitude of the vector $\mathbf{v}$. In other words, $W(x, y)$ is constant along ellipses

$$\frac{x^2}{t_x} + \frac{y^2}{t_y} = \text{constant}.$$

(c) How does the average number of steps, $\langle \ell \rangle = \langle \ell_x + \ell_y \rangle$, diverge as $\bar{t}$ approaches t_c ?

- The weight of all walks of length ℓ , irrespective of their end point location, is

$$\sum_{x,y} \langle 0, 0 | W(\ell) | x, y \rangle = \langle 0, 0 | T^\ell | q_x = 0, q_y = 0 \rangle = (2t_x + 2t_y)^\ell = (4\bar{t})^\ell.$$

Therefore,

$$\langle \ell \rangle = \frac{\sum_{\ell} \ell (4\bar{t})^\ell}{\sum_{\ell} (4\bar{t})^\ell} = 4\bar{t} \frac{\partial}{\partial (4\bar{t})} \ln \left[\sum_{\ell} (4\bar{t})^\ell \right] = 4\bar{t} \frac{\partial}{\partial (4\bar{t})} \ln \frac{1}{1 - 4\bar{t}} = \frac{4\bar{t}}{1 - 4\bar{t}},$$

i.e.

$$\langle \ell \rangle = \frac{\bar{t}}{t_c - \bar{t}},$$

diverges linearly close to the singular value of $\bar{t}$.

8. Anisotropic Ising Model: Consider the anisotropic Ising model on a square lattice with a Hamiltonian

$$-\beta \mathcal{H} = \sum_{x,y} (K_x \sigma_{x,y} \sigma_{x+1,y} + K_y \sigma_{x,y} \sigma_{x,y+1});$$

i.e. with bonds of different strengths along the x and y directions.

(a) By following the method presented in the text, calculate the free energy for this model. You do not have to write down every step of the derivation. Just sketch the steps that need to be modified due to anisotropy; and calculate the final answer for $\ln Z/N$.

- The Hamiltonian

$$-\beta \mathcal{H} = \sum_{x,y} (K_x \sigma_{x,y} \sigma_{x+1,y} + K_y \sigma_{x,y} \sigma_{x,y+1}),$$

leads to

$$Z = \sum (2 \cosh K_x \cosh K_y)^N t_x^{\ell_x} t_y^{\ell_y},$$

where $t_i = \tanh K_i$, and the sum runs over all closed graphs. By extension of the isotropic case,

$$f = \frac{\ln Z}{N} = \ln (2 \cosh K_x \cosh K_y) + \sum_{\ell_x, \ell_y} \frac{t_x^{\ell_x} t_y^{\ell_y}}{\ell_x + \ell_y} \langle 0 | W^* (\ell_x, \ell_y) | 0 \rangle,$$

where

$$\langle 0 | W^* (\ell_x, \ell_y) | 0 \rangle = \frac{1}{2} \sum' (-1)^{\text{number of crossings}},$$

and the primed sum runs over all directed (ℓ_x, ℓ_y) -steps walks from $(0,0)$ to $(0,0)$ with no U-turns. As in the isotropic case, this is evaluated by taking the trace of powers of the $4N \times 4N$ matrix described by Eq.(VI.66), which is block-diagonalized by Fourier transformation. However, unlike the isotropic case, in which each element is multiplied by t , here they are multiplied by t_x and t_y , respectively, resulting in

$$f = \ln (2 \cosh K_x \cosh K_y) + \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \text{tr} \ln [1 - T(\mathbf{q})^*],$$

where

$$\begin{aligned} \text{tr} \ln [1 - T(\mathbf{q})^*] &= \ln \det [1 - T(\mathbf{q})^*] \\ &= \ln [(1 + t_x^2)(1 + t_y^2) - 2t_x(1 - t_y^2) \cos q_x - 2t_y(1 - t_x^2) \cos q_y] \\ &= \ln \left[\frac{\cosh 2K_x \cosh 2K_y - \sinh 2K_x \cos q_x - \sinh 2K_y \cos q_y}{\cosh^2 K_x \cosh^2 K_y} \right], \end{aligned}$$

resulting in

$$f = \ln 2 + \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \ln (\cosh 2K_x \cosh 2K_y - \sinh 2K_x \cos q_x - \sinh 2K_y \cos q_y).$$

(b) Find the critical boundary in the (K_x, K_y) plane from the singularity of the free energy. Show that it coincides with the condition $K_x = \tilde{K}_y$, where $\tilde{K}$ indicates the standard dual interaction to K .

- The argument of the logarithm is minimal at $q_x = q_y = 0$, and equal to

$$\begin{aligned} &\cosh 2K_x \cosh 2K_y - \sinh 2K_x - \sinh 2K_y \\ &= \frac{1}{2} \left(e^{K_x} \sqrt{\cosh 2K_y - 1} - e^{-K_x} \sqrt{\cosh 2K_y + 1} \right)^2. \end{aligned}$$

Therefore, the critical line is given by

$$e^{2K_x} = \sqrt{\frac{\cosh 2K_y + 1}{\cosh 2K_y - 1}} = \coth K_y.$$

Note that this condition can be rewritten as

$$\sinh 2K_x = \frac{1}{2} (\coth K_y - \tanh K_y) = \frac{1}{\sinh 2K_y},$$

i.e. the critical boundary can be described as $K_x = \tilde{K}_y$, where the dual interactions, $\tilde{K}$ and K , are related by $\sinh 2K \sinh 2\tilde{K} = 1$.

(c) Find the singular part of $\ln Z/N$, and comment on how anisotropy affects critical behavior in the exponent and amplitude ratios.

• The singular part of $\ln Z/N$ for the anisotropic case can be written as

$$f_S = \frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} \ln \left[\left(e^{K_x} \sqrt{\cosh 2K_y - 1} - e^{-K_x} \sqrt{\cosh 2K_y + 1} \right)^2 + \sum_{i=x,y} \frac{q_i^2}{2} \sinh 2K_i \right].$$

In order to rewrite this expression in a form closer to that of the singular part of the free energy in the isotropic case, let

$$q_i = \sqrt{\frac{2}{\sinh 2K_i}} q'_i,$$

and

$$\delta t = e^{K_x} \sqrt{\cosh 2K_y - 1} - e^{-K_x} \sqrt{\cosh 2K_y + 1}$$

(δt goes linearly through zero as (K_x, K_y) follows a curve which intersects the critical boundary). Then

$$f_S = \frac{1}{\sqrt{\sinh 2K_x \sinh 2K_y}} \int \frac{d^2 q'}{(2\pi)^2} \ln (\delta t^2 + q'^2).$$

Thus, upon approaching the critical boundary ($\sinh 2K_x \sinh 2K_y = 1$), the singular part of the anisotropic free energy coincides more and more precisely with the isotropic one, and the exponents and amplitude ratios are unchanged by the anisotropy. (The amplitudes themselves obviously depend on the location)

9. Müller–Hartmann Zittartz estimate of the interfacial energy of the $d = 2$ Ising model on a square lattice:

(a) Consider an interface on the square lattice with periodic boundary conditions in one direction. Ignoring islands and overhangs, the configurations can be labelled by heights h_n for $1 \leq n \leq L$. Show that for an anisotropic Ising model of interactions (K_x, K_y) , the energy of an interface along the x -direction is

$$-\beta \mathcal{H} = -2K_y L - 2K_x \sum_n |h_{n+1} - h_n|.$$

is associated the eigenvalue

$$\lambda_k = e^{-2K_y} \sum_{n=1}^{H+1} T_{1n} e^{i \frac{2\pi}{k} \cdot (n-1)}.$$

Note that there are $H + 1$ eigenvectors, corresponding to $k = 1, \dots, H + 1$.

(c) Obtain the interface free energy using the result in (b), or by any other method.

• One way of obtaining the free energy is to evaluate the largest eigenvalue of T . Since all elements of T are positive, the eigenvector $(1, 1, \dots, 1)$ has the largest eigenvalue

$$\begin{aligned} \lambda_1 &= e^{-2K_y} \sum_{n=1}^{H+1} T_{1n} = e^{-2K_y} \left(1 + 2 \sum_{n=1}^{H/2} e^{-2K_x n} \right) \\ &= e^{-2K_y} \left(2 \sum_{n=0}^{H/2} e^{-2K_x n} - 1 \right) = e^{-2K_y} \coth K_x, \end{aligned}$$

in the $H \rightarrow \infty$ limit. Then, $F = -Lk_B T \ln \lambda_1$.

Alternatively, we can directly sum the partition function, as

$$\begin{aligned} Z &= e^{-2K_y L} \sum_{\{h_n\}} \exp \left(-2K_x \sum_{n=1}^L |h_{n+1} - h_n| \right) = e^{-2K_y L} \left[\sum_d \exp(-2K_x |d|) \right]^L \\ &= \left[e^{-2K_y} \left(2 \sum_{d \geq 0} e^{-2K_x d} - 1 \right) \right]^L = (e^{-2K_y} \coth K_x)^L, \end{aligned}$$

yielding

$$F = -Lk_B T [\ln (\coth K_x) - 2K_y].$$

(d) Find the condition between K_x and K_y for which the interfacial free energy vanishes. Does this correspond to the critical boundary of the original 2d Ising model?

• The interfacial free energy vanishes for

$$\coth K_x = e^{2K_y},$$

which coincides with the result from an earlier problem. This illustrates that long wavelength fluctuations, such as interfaces, are responsible for destroying order at criticality.

10. Anisotropic Landau Theory: Consider an n -component magnetization field $\vec{m}(\mathbf{x})$ in d -dimensions.

(a) Using the previous problems on anisotropy as a guide, generalize the standard Landau–Ginzburg Hamiltonian to include the effects of spacial anisotropy.

- Requiring different coupling constants in the different spatial directions, along with rotational invariance in spin space, leads to the following leading terms of the Hamiltonian,

$$-\beta\mathcal{H} = \int d^d x \left[\frac{t}{2} \vec{m}(\mathbf{x})^2 + \sum_{i=1}^d \frac{K_i}{2} \frac{\partial \vec{m}}{\partial x_i} \cdot \frac{\partial \vec{m}}{\partial x_i} + u \vec{m}(\mathbf{x})^4 \right].$$

(b) Are such anisotropies “relevant?”

- Clearly, the apparent anisotropy can be eliminated by the rescaling

$$x'_i = \sqrt{\frac{K}{K_i}} x_i.$$

In terms of the primed space variables, the Hamiltonian is isotropic. In particular, the universal features are identical in the anisotropic and isotropic cases, and the anisotropy is thus “irrelevant” (provided all K_i are non-vanishing).

(c) In La_2CuO_4 , the Cu atoms are arranged on the sites of a square lattice in planes, and the planes are then stacked together. Each Cu atom carries a “spin”, which we assume to be classical, and can point along any direction in space. There is a very strong antiferromagnetic interaction in each plane. There is also a very weak interplane interaction that prefers to align successive layers. Sketch the low-temperature magnetic phase, and indicate to what universality class the order–disorder transition belongs.

- For classical spins, this combination of antiferromagnetic and ferromagnetic couplings is equivalent to a purely ferromagnetic (anisotropic) system, since we can redefine (*e.g.* in the partition function) all the spins on one of the two sublattices with an opposite sign. Therefore, the critical behavior belongs to the $d = 3$, $n = 3$ universality class.

Nevertheless, there is a range of temperatures for which the in-plane correlation length is large compared to the lattice spacing, while the interplane correlation length is of the order of the lattice spacing. The behavior of the system is then well described by a $d = 2$, $n = 3$ theory.
