Spin glass

1. *Mean-Field Edwards–Anderson Model with Ferromagnetic Bias:* Consider the Hamiltonian

$$H = -\sum_{<i,j>} J_{ij}\sigma_i\sigma_j - h_0\sum_i \sigma_i\,,$$

where each spin has z neighbors. The interactions $\{J_{ij}\}$ are independently distributed with mean $\overline{J}/z$ and variance J^2/z.

(a) Within the mean field approximation, what is the distribution of the effective field $\tilde{h}$?

(b) Show that the self-consistent equation for $m = \overline{\langle\sigma\rangle}$ takes the form

$$m = \int dx \frac{e^{-x^2/2}}{\sqrt{2\pi}} \tanh[\beta(h_0 + \overline{J}m + J\sqrt{q}x)]\,,$$

and derive the corresponding equation for $q = \overline{\langle\sigma\rangle^2}$.

(c) At the disordered to glass transition, q becomes non-zero, while m stays zero. Find the transition temperature $T_c = 1/\beta_c$, and the critical behaviors of q with $(T_c - T)$ on the ordered side.

(d) At the disordered to ferromagnetic transition m and q switch from zero to non-zero values. Find the transition temperature $T_c = 1/\beta_c$, and the critical behaviors of m and q with $(T_c - T)$ on the ordered side.

(e) At the spin glass to ferromagnetic transition m switches from zero to non-zero value at a finite value of q. Find the transition temperature $T_c = 1/\beta_c$, and the critical behaviors of m and q with $(T_c - T)$ on the ordered side.

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2. *Replica analysis of the Sherrington–Kirkpatrick model Model with Ferromagnetic Bias* Consider the Hamiltonian

$$H = -\sum_{i<j=1}^{N} J_{ij}\sigma_i\sigma_j - h_0\sum_i \sigma_i\,,$$

where each spin is coupled to all other $(N-1)\approx N$ spins with interaction $\{J_{ij}$ that are independently distributed with $\langle J_{ij}\rangle = \overline{J}/N$ and variance $\langle J_{ij}^2\rangle_c = J^2/N$.

(a) Perform the disorder average of $\overline{Z^n}$ using the replica trick. Show that in addition to the quadratic term in $Q_{\alpha\beta}$, a linear coupling to $\sum_\alpha m_\alpha$ appears, where $m_\alpha = \frac{1}{N}\sum_i \sigma_i^\alpha$.

(b) Using the replica symmetric (RS) ansatz $Q_{\alpha\beta} = q$ for $\alpha \neq \beta$ and $m_\alpha = m$ for all α, find the effective free energy $f(m, q)$.

(c) Derive the coupled saddle-point equations for m and q.

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3. (Optional:) *Instability of the mean field solution:* For this problem set $\overline{J} = 0$, and (except in calculating susceptibility) $h_0 = 0$.

(a) Show that the zero-field susceptibility satisfies $\chi = \beta(1 - q)$.

(b) Using the RS expression for free energy, compute the internal energy E in the limit $h_0 = \overline{J} = 0$.

(c) Show that as $T = 1/\beta \to 0$, $q \approx 1 - 4/(\sqrt{2\pi}T/J$.

(d) Show that the RS energy goes down linearly with T for $T > T_c$, and that it attains a value $E(T = 0) > J$, which must imply a non-monotonic form of the RS energy as function of temperature, and hence a thermodynamic instability.

(e) By considering small fluctuations around the RS solution in the overlap matrix $Q_{\alpha\beta}$, it can be shown that the most important eigenvalue of the stability matrix in the $n \to 0$ limit, is the so-called replicon eigenvalue

$$\lambda_{\mathrm{R}} = 1 - \beta^2 J^2 \int Dx \, \mathrm{sech}^4(\beta J \sqrt{q}\, x).$$

Show that this implies that the RS solution is always unstable.

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