Scaling, Renormalization, & Anisotropy

1. *Corrections to scaling:* While relevant operators (renormalization group eigendirections with positive eigenvalues) are responsible for the leading singularities close to a critical point, irrelevant operators (with negative eigenvalues) lead to corrections to scaling that can be observed in experiments and simulations. Consider the singular homogeneous free energy with scaling form

$$f(t,g) = b^{-d} f(b^{y_t} t, b^{y_g} g)\,,$$

with $y_t > 0$ and $y_g < 0$.

(a) Choose b to manifest the leading singularity as function of t. Assuming that the resulting function can be expanded analytically for small argument, find the next to leading singularity in t.

(b) The heat capacity observed in an experiment (in $d = 3$) is fitted to $C = t^{-0.1}[A + B\sqrt{t}]$. What do you conclude about y_t and y_g?

(c) Divergence of the heat capacity is a reflection of the power-law decay of energy density correlations $\langle E(\mathbf{x})E(\mathbf{0})\rangle \propto |x|^{-2\Delta_E}$ at the critical point. How is the leading decay exponent of $2\Delta_E$ related to the exponent y_t?

(d) How are the corrections to scaling due to the irrelevant operator g manifested at the critical point?

****

2. (Optional) *Finite size scaling:* Numerical simulations are a common tool for estimating critical exponents. However, thermodynamic singularities and divergences occur only in the limit of an infinite system. For a system of finite size L^d, thermodynamic functions are analytic, and divergences are cutoff when the correlation length ξ becomes of the order of the system size L.

(a) Assuming a divergent heat capacity ($\alpha > 0$) how does the maximum heat capacity observed in a finite sample depend on its size L?

(b) The procedure of *data collapse* provides a powerful method to validate scaling predictions, such as

$$C[(T-T_c), L] = b^{d/y_t - 2} C[b^{y_t}(T-T_c), L/b]\,.$$

Describe how one would collapse data for different sizes L to determine T_c and the exponent $\nu = 1/y_t$.

****

3. *Dangerously irrelevant:* For $d > 4$ dimensions, the singular part of the Landau–Ginzburg free energy has the homogeneous form

$$f(t, h, u) = b^{-d} f(b^2 t, b^{1+d/2} h, b^{4-d} u)\,.$$

While the parameter u is technically irrelevant, for $t < 0$ its inclusion is necessary to keep the magnetization finite.

(a) For $t > 0$, an analytic expansion in u is possible, and generates *sub-leading corrections to scaling.* By considering the term in f proportional to uh^2, find the leading correction to (zero field) susceptibility that is proportional to u.

(b) For $t < 0$, a finite $u > 0$ is necessary for a well defined model, and while formally irrelevant, its importance is manifested in generating *non-analytic terms* in the expansion of free energy, such as proportional to u^{-1} and $u^{-1/2}|h|$. Find the $(-t)$ dependence of these terms, and give their physical meaning.

****

4. *Coupled scalars:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 - hm + \frac{L}{2} (\nabla^2 \phi)^2 + v\; m(\nabla^2 \phi) \right],$$

coupling two scalar fields m and ϕ. Note that the included terms (with the exception of the symmetry breaking field h) satisfy the symmetries $(m, \phi) \to (-m, -\phi)$, and $\nabla\phi \to \nabla\phi + \vec{c}$.

(a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\mathbf{q})$ and $\phi(\mathbf{q})$.

(b) Construct a renormalization group transformation as in class, by rescaling distances such that $\mathbf{q}' = b\mathbf{q}$; and the fields such that $m'(\mathbf{q}') = \tilde{m}(\mathbf{q})/z$ and $\phi'(\mathbf{q}') = \tilde{\phi}(\mathbf{q})/y$. Do not evaluate the integrals that just contribute a constant additive term.

(c) There is a fixed point such that $K' = K$ and $L' = L$. Find y_t, y_h and y_v at this fixed point.

(d) The singular part of the free energy has a scaling from $f(t,h,v) = t^{2-\alpha} g(h/t^{\Delta}, v/t^{\omega})$ for t, h, v close to zero. Find α, Δ, and ω.

(e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point, and how do they scale?

(f) What are the *lowest order* nonlinearities consistent with the symmetries indicated?

****

5. *Long–range interactions* between spins can be described by adding a term

$$\int d^d\mathbf{x} \int d^d\mathbf{y} J(|\mathbf{x}-\mathbf{y}|)\vec{m}(\mathbf{x}) \cdot \vec{m}(\mathbf{y}),$$

to the usual Landau–Ginzburg Hamiltonian.

(a) Show that for $J(r) \propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\beta\mathcal{H} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{t + K_2 q^2 + K_\sigma q^\sigma + \cdots}{2} \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q})$$
$$+ u \int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}} \vec{m}(\mathbf{q}_1) \cdot \vec{m}(\mathbf{q}_2)\, \vec{m}(\mathbf{q}_3) \cdot \vec{m}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \quad .$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν, η, and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

(e) For $\sigma < 2$, use a perturbation expansion in u to construct the recursion relations for (t, K_2, u) as in the text.

(f) For $d < 2\sigma$, calculate the critical exponents ν and η to first order in $\epsilon = 2\sigma - d$.

[See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).]

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!

****

6. (Optional) *Smectic liquid crystal* is an anisotropic form of matter which can be imagined as two dimensional layers stacked along a third (say z) dimension. The periodically varying density can be described by $\rho(z, \mathbf{x}) = \rho_0 + \overline{\psi} \cos[kz + u(z, \mathbf{x})]$, with $a = 2\pi/k$ indicating the distance between layers. Distortions from a perfect stacking $u = 0$, are described by the spatially varying field $u(z, \mathbf{x})$. Due to the underlying anisotropy, the cost of distortions is governed by the Hamiltonian

$$\beta\mathcal{H} = \frac{1}{2} \int dz d^2\mathbf{x} \left[B(\partial_z u)^2 + K_1 (\partial_{\mathbf{x}}^2 u)^2 \right] .$$

The origin of such anisotropic Goldstone modes is explored in this problem.

(a) Consider a complex field $\psi(\vec{x})$ subject to the effective Hamiltonian

$$\beta\mathcal{H} = \int d^d\vec{x} \left[\frac{t}{2}|\psi|^2 + u|\psi|^4 + \frac{G}{2}\left(|\nabla\psi|^2 - k^2|\psi|^2 \right)^2 + \frac{L}{2}|\nabla^2\psi|^2 \right] .$$

Note that to prevent the instability arising from a negative coefficient $(-Gk^2)$ for $|\nabla\psi|^2$, a positive higher order term $G|\nabla\psi|^4$ is introduced. Find the most probable (saddle point) configuration $\overline{\psi}(\vec{x})$.

(b) Include fluctuations by setting $\psi(\vec{x}) = \overline{\psi} e^{i[kz+u(z,\mathbf{x})]}$, and including only leading terms in the gradient expansion.

(c) Identify the moduli B and K_1 of the smectic liquid crystal, in term of the parameters of the above model.

****

7. (Optional) *Anisotropic criticality:* A number of materials, such as smectic liquid crystals, are anisotropic and behave differently along distinct directions, which shall be denoted parallel and perpendicular, respectively. Let us assume that the d spatial dimensions are grouped into n parallel directions $\mathbf{x}_\parallel$, and $d - n$ perpendicular directions $\mathbf{x}_\perp$. Consider a one–component field $m(\mathbf{x}_\parallel, \mathbf{x}_\perp)$ subject to a Landau–Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int d^n\mathbf{x}_\parallel d^{d-n}\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm \right] ,$$

and

$$U = u \int d^n\mathbf{x}_\parallel d^{d-n}\mathbf{x}_\perp \; m^4 .$$

(Note that $\beta\mathcal{H}$ depends on the **first** *gradient in the $\mathbf{x}_\parallel$ directions, and on the* **second** *gradient in the $\mathbf{x}_\perp$ directions.)*

(a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(\mathbf{q}_{\|}, \mathbf{q}_{\perp})$.

(b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $\mathbf{q}'_{\|} = b\ \mathbf{q}_{\|}$ and $\mathbf{q}'_{\perp} = c\ \mathbf{q}_{\perp}$ and the field as $m'(\mathbf{q}') = m(\mathbf{q})/z$. *Note that parallel and perpendicular directions are scaled differently.* Write down the recursion relations for K, L, t, and h in terms of b, c, and z. (The exact shape of the Brillouin zone is immaterial at this stage, and you do not need to evaluate the integral that contributes an additive constant.)

(c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h for the rescalings of t and h.

(d) Write the relationship between the (singular parts of) free energies $f(t, h)$ and $f'(t', h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t, h) = t^{2-\alpha} g_f(h/t^{\Delta})$, and identify the exponents α and Δ.

(e) How does the unperturbed zero–field susceptibility $\chi(t, h = 0)$, diverge as $t \to 0$?

In the remainder of this problem set $h = 0$, and treat U as a perturbation.

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(q)m(q')\rangle_0$, and the corresponding susceptibility $\chi_0(q) \propto \langle |m_q|^2\rangle_0$, where q stands for $(\mathbf{q}_{\|}, \mathbf{q}_{\perp})$.

(g) Write the perturbation U, in terms of the normal modes $m(q)$.

(h) Using RG, or any other method, find the upper critical dimension d_u, for validity of the Gaussian exponents.

(i) Write down the expansion for $\langle m(q)m(q')\rangle$, to first order in U, and reduce the correction term to a product of two point expectation values.

(j) Write down the expression for $\chi(q)$, in first order perturbation theory, and identify the transition point t_c at order of u. (Do not evaluate any integrals explicitly.)

****

8. *Percolation:* The order parameter of percolation is a probability and thus strictly non-negative. The percolation transition can be modeled by a statistical field theory in terms of a scalar field $m(\mathbf{x})$ *which is strictly non-negative at all points*, and subject to weight governed by the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + \frac{c}{3}m^3 - hm\right],$$

with $(K, c) > 0$ for stability, and $h \geq 0$ included to bias for $m(\mathbf{x}) \geq 0$. In problem set #1 we introduced the mean-field theory for the q-state Potts model which resembles the above expression for $q < 2$. Indeed, there is a rigorous relation between percolation, and the $q \to 1$ limit of Potts models.

(a) In the saddle point approximation find the most probable state $m(\mathbf{x})$; the exponent β governing the vanishing of the order parameter with t, and the gap exponent Δ.

(b) A corresponding free energy is obtained as $f(t) = -\ln Z/V$, where the normalization $Z(t)$ is obtained by integrating $\exp(-\beta\mathcal{H})$ over all configuration of $m(\mathbf{x})$. What is the form of the singularity of $f(t, h \to 0)$ in the saddle point approximation?

(c) Without detailed calculations, can you identify the upper critical dimension for validity of saddle point results?

****