8.334: Statistical Mechanics II — Problem Set # **2** — Due: **3/6/25**

Fluctuations

1. *The Higgs mechanism:* The Hamiltonian for a superconductor in the presence of a magnetic field takes the form

$$\beta\mathcal{H} = \int d^3\vec{x}\left[\frac{t}{2}|\psi|^2 + u|\psi|^4 + \frac{K}{2}|\vec{\nabla}\psi - ie\vec{A}\,\psi|^2 + \frac{L}{2}(\vec{\nabla}\times\vec{A})^2\right],$$

where the complex field $\psi(\vec{x})$ is the superconducting order parameter, $\vec{A}(\vec{x})$ is the electromagnetic gauge field, such that $\vec{B} = \vec{\nabla}\times\vec{A}$ is the magnetic field, and K, L, and u are positive. Note that the joint (gauge) symmetry $\psi \to \psi e^{ie\phi(\vec{x})}$ and $\vec{A} \to \vec{A} + \vec{\nabla}\phi(\vec{x})$ leaves the Hamiltonian unchanged.

(a) The guage symmetry allows us to set $\vec{\nabla}\cdot\vec{A} = 0$ (with choice of $e\vec{\nabla}^2\phi = \vec{\nabla}\cdot\vec{A}$). Justify why $(\vec{\nabla}\times\vec{A})^2$ can then be replaced by $(\nabla\vec{A})^2 \equiv \sum_{\alpha,\beta}\partial_\alpha A_\beta \partial_\alpha A_\beta$. Assuming a corresponding replacement is possible in all dimensions, in the remainder of the problem generalize from 3 to arbitrary dimensions d.

(b) Show that there is a saddle point solution of the form $\psi(\vec{x}) = \overline{\psi}e^{ie\phi(\vec{x})}$ and $A(\vec{x}) = \vec{\nabla}\phi(\vec{x})$, and find $\overline{\psi}$ for $t > 0$ and $t < 0$.

(c) Sketch the heat capacity $C = \partial^2 \ln Z/\partial t^2$, and discuss its singularity as $t \to 0$ in the saddle point approximation.

(d) Include fluctuations by setting

$$\begin{cases} \psi(\mathbf{x}) = \left(\overline{\psi} + \psi_\ell(\mathbf{x})\right)e^{ie\phi(\mathbf{x})} \\ \vec{A}(\mathbf{x}) = \vec{\nabla}\phi(\mathbf{x}) + \vec{a}(\mathbf{x}) \end{cases}.$$

and expanding $\beta\mathcal{H}$ to quadratic order in ψ_ℓ and $\vec{a}$.

(e) Find the correlation length for the longitudinal fluctuations ξ_ℓ, for $t > 0$ and $t < 0$.

(f) Find the correlation length ξ_a for the fluctuations of the 'transverse' field $\vec{a}$, for $t > 0$ and $t < 0$. (Note that the field $\vec{A}$ acquires a correlation length (mass) due to spontaneous symmetry breaking of the (Higgs) field ψ.)

(g) Calculate the correlation function $\langle a_i(\mathbf{x})a_j(\mathbf{0})\rangle$ for $t > 0$.

(h) Compute the correction to the saddle point free energy $\ln Z$, from fluctuations. (You can leave the answer in the form of integrals involving ξ_ℓ and ξ_a.)

(i) Find the fluctuation corrections to the heat capacity in (b), again leaving the answer in the form of integrals.

(j) Discuss the behavior of the integrals appearing above schematically, and state their dependence on the correlation length ξ, and cutoff Λ, in different dimensions.

(k) What is the critical dimension for the validity of saddle point results, and how is it modified by the coupling to the vector field $\vec{A}$?

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2. *Random magnetic fields:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2} (\nabla m)^2 + \frac{t}{2} m^2 + u\, m^4 - h(\mathbf{x}) m(\mathbf{x}) \right] \quad ,$$

where $m(\mathbf{x})$ and $h(\mathbf{x})$ are scalar fields, and $u > 0$. The random magnetic field $h(\mathbf{x})$ results from frozen (quenched) impurities that are independently distributed in space. For simplicity $h(\mathbf{x})$ is assumed to be an independent Gaussian variable at each point $\mathbf{x}$, such that

$$\overline{h(\mathbf{x})} = 0, \qquad \text{and} \qquad \overline{h(\mathbf{x})h(\mathbf{x}')} = \Delta\delta^d(\mathbf{x} - \mathbf{x}'), \tag{1}$$

where the over-line indicates (*quench*) averaging over all values of the random fields. The above equation implies that the Fourier transformed random field $h(\mathbf{q})$ satisfies

$$\overline{h(\mathbf{q})} = 0, \qquad \text{and} \qquad \overline{h(\mathbf{q})h(\mathbf{q}')} = \Delta(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}'). \tag{2}$$

(a) Calculate the quench averaged free energy, $\overline{f_{sp}} = \overline{\min\{\Psi(m)\}_m}$, *assuming a saddle point solution with uniform magnetization* $m(\mathbf{x}) = m$. (Note that with this assumption, the random field disappears as a result of averaging and has no effect at this stage.)

(b) Include fluctuations by setting $m(\mathbf{x}) = \overline{m} + \phi(\mathbf{x})$, and expanding $\beta\mathcal{H}$ to second order in ϕ.

(c) Express the energy cost of the above fluctuations in terms of the Fourier modes $\phi(\mathbf{q})$.

(d) Calculate the mean $\langle\phi(\mathbf{q})\rangle$, and the variance $\left\langle|\phi(\mathbf{q})|^2\right\rangle_c$, where $\langle\cdots\rangle$ denotes the usual thermal expectation value *for a fixed* $h(\mathbf{q})$.

(e) Use the above results, in conjunction with Eq.(2), to calculate the quench averaged scattering line shape $S(q) = \overline{\langle|\phi(\mathbf{q})|^2\rangle}$.

(f) Perform the Gaussian integrals over $\phi(\mathbf{q})$ to calculate the fluctuation corrections, $\delta f[h(\mathbf{q})]$, to the free energy.

$$\left(\text{Reminder :} \qquad \int_{-\infty}^{\infty} d\phi d\phi^* \exp\left(-\frac{K}{2}|\phi|^2 + h^*\phi + h\phi^*\right) = \frac{2\pi}{K}\exp\left(\frac{2|h|^2}{K}\right) \qquad \right)$$

(g) Use Eq.(2) to calculate the corrections due to the fluctuations in the previous part to the quench averaged free energy $\overline{f}$. (Leave the corrections in the form of two integrals.)

(h) Estimate the singular t dependence of the integrals obtained in the fluctuation corrections to the free energy.

(i) Find the upper critical dimension, d_u, for the validity of saddle point critical behavior.

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3. *Long–range interactions:* Consider a continuous spin field $\vec{s}(\mathbf{x})$, subject to a long–range ferromagnetic interaction

$$\int d^d\mathbf{x} d^d\mathbf{y} \frac{\vec{s}(\mathbf{x}) \cdot \vec{s}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}},$$

as well as short-range interactions.

(a) How is the quadratic term in the Landau-Ginzburg expansion modified by the presence of this long-range interaction? For what values of σ is the long-range interaction dominant?

(b) By evaluating the magnitude of thermally excited Goldstone modes (or otherwise), obtain the lower critical dimension d_ℓ below which there is no long–range order.

(c) Find the upper critical dimension d_u, above which saddle point results provide a correct description of the phase transition.

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4. *Crumpling:* Configurations of a two-dimensional sheet are described by a vector field $\vec{r}(\mathbf{x}) = (r_1, r_2, \cdots, r_d)$, denoting the embedding of an element $\mathbf{x} = (x_1, x_2)$ in d-dimensional space. The energy cost of such configuration is described by

$$\beta\mathcal{H}[\vec{r}] = \int d^2\mathbf{x} \left[\frac{t}{2}\partial_\alpha r_i \partial_\alpha r_i + \frac{K}{2}\partial_\alpha\partial_\alpha r_i \partial_\beta\partial_\beta r_i + u\partial_\alpha r_i\partial_\alpha r_i\partial_\beta r_j\partial_\beta r_j + v\partial_\alpha r_i\partial_\beta r_i\partial_\alpha r_j\partial_\beta r_j\right],$$

with implicit summation over $\alpha, \beta = 1, 2$ and $i, j = 1, 2, \cdots d$.

(a) For $t < 0$ (with positive K) the most likely configuration of the sheet is stretched along a spontaneously chosen direction. Setting $\partial_\alpha r_i = m\delta_{\alpha,i}$ (noting $\alpha = 1, 2$, while $i = 1, 2, 3$), find the most likely value of m.

(b) Include fluctuations by setting

$$\begin{cases} r_\alpha(\mathbf{x}) = m\big(x_\alpha + u_\alpha(\mathbf{x})\big) & \text{for } \alpha = 1, 2 \\ r_i(\mathbf{x}) = m\ h_i(\mathbf{x}) & \text{for } i = 3, \cdots, d \end{cases} .$$

and expanding $\beta\mathcal{H}$ to quadratic order in u_i and h, *and to lowest power in gradients.*

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5. (Optional) *Ginzburg criterion along the field direction:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 - \vec{h}\cdot\vec{m}\right] ,$$

describing an n–component magnetization vector $\vec{m}(\mathbf{x})$, with $u > 0$.

(a) In the saddle point approximation, the free energy is $f = \min\{\Psi(m)\}_m$. Sketch the form of the resulting magnetization isotherms $\overline{m}(h, t)$ for $t > 0$, $t = 0$, and $t < 0$, and the corresponding phase boundary in the (h, t) plane. (h denotes the magnitude of $\vec{h}$.)

(b) For t and h close to zero, the magnetization has the scaling form $\overline{m} = t^\beta g_m(h/t^\Delta)$. Identify the exponents β and Δ in the saddle point approximation.

For the remainder of this problem set $t = 0$.

(c) Include fluctuations by setting $\vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expanding $\beta\mathcal{H}$ to second order in the ϕs. ($\hat{e}_\ell$ is a unit vector parallel to the average magnetization, and $\hat{e}_t$ is perpendicular to it.)

(d) Calculate the longitudinal and transverse correlation lengths, ξ_ℓ and ξ_t.

(e) Calculate the first correction to the free energy from these fluctuations. You don't have to evaluate any integrals- just use dimensional arguments to express the singular part of the correction in terms of scaling forms involving the correlation length $\xi_\ell \propto \xi_t$.

(f) Using the above singular scaling form find the fluctuation–correction to magnetization, and obtain an upper critical dimension by comparison to the saddle–point value.

(g) For $d < d_u$ obtain a Ginzburg criterion by finding the field h_G below which fluctuations are important. (You may ignore the numerical coefficients in h_G, but the dependances on K and u are required.)

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