

III.F The Gaussian Model (Direct solution)

The RG approach will be applied to the *Gaussian model* in the next section. For the sake of later comparison, here we provide the direct solution of this problem. The Gaussian model is obtained by keeping only the quadratic terms in the Landau–Ginzburg expansion. The resulting partition function is

$$Z = \int \mathcal{D}\vec{m}(\mathbf{x}) \exp \left\{ - \int d^d \mathbf{x} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 + \frac{L}{2} (\nabla^2 m)^2 + \cdots - \vec{h} \cdot \vec{m} \right] \right\}. \quad (\text{III.49})$$

Clearly the model is well defined only for $t \geq 0$, since there is no m^4 term to insure its stability for $t < 0$. The partition function still has a singularity at $t = 0$, and we can regard this as representing approaching a phase transition from the disordered side.

The quadratic form is easily evaluated by the usual rules of Gaussian integration. The kernel is first diagonalized by the Fourier modes: The allowed $\mathbf{q}$ values are discretized in a finite system of size L , with spacing of $2\pi/L$. The largest $\mathbf{q}$ are limited by the lattice spacing, and confined to a *Brillouin Zone* whose shape is determined by the underlying lattice. We shall in fact use a slightly different normalization for the Fourier modes, and keep careful track of the volume factors, by setting

$$\begin{cases} \vec{m}(\mathbf{q}) = \int d^d \mathbf{x} e^{i\mathbf{q} \cdot \mathbf{x}} \vec{m}(\mathbf{x}) \\ \vec{m}(\mathbf{x}) = \sum_{\mathbf{q}} \frac{e^{-i\mathbf{q} \cdot \mathbf{x}}}{V} \vec{m}(\mathbf{q}) = \int \frac{d^d \mathbf{q}}{(2\pi)^d} e^{-i\mathbf{q} \cdot \mathbf{x}} \vec{m}(\mathbf{q}) \end{cases}. \quad (\text{III.50})$$

(We should really use a different symbol, such as $\tilde{m}_i(\mathbf{q})$ to indicate the Fourier modes. For the sake of brevity we use the same symbol, but explicitly include the argument $\mathbf{q}$ as the indicator of the Fourier transformed function.) The last transformation applies to the infinite size limit ($L \rightarrow \infty$), and V is the system volume.

In reexpressing the Hamiltonian in terms of Fourier modes, we encounter expressions such as

$$\int d^d \mathbf{x} m(\mathbf{x})^2 = \int d^d \mathbf{x} \sum_{\mathbf{q}, \mathbf{q}'} \frac{e^{-i(\mathbf{q} + \mathbf{q}') \cdot \mathbf{x}}}{V^2} \vec{m}(\mathbf{q}) \cdot \vec{m}(\mathbf{q}') = \sum_{\mathbf{q}} \frac{\vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q})}{V}. \quad (\text{III.51})$$

The last expression follows from the vanishing of the integral over $\mathbf{x}$ unless $\mathbf{q} + \mathbf{q}' = \mathbf{0}$, in which case it equals the system volume. Similar manipulations leads to the Hamiltonian

$$\beta \mathcal{H} = \sum_{\mathbf{q}} \left(\frac{t + Kq^2 + Lq^4 + \cdots}{2V} \right) |\vec{m}(\mathbf{q})|^2 - \vec{h} \cdot \vec{m}(\mathbf{q} = \mathbf{0}). \quad (\text{III.52})$$

With the choice of the normalization in eq.(III.50), the Jacobian of the transformation to Fourier modes is $1/\sqrt{V}$ per mode, and the partition function equals

$$Z = \prod_{\mathbf{q}} V^{-n/2} \int d\vec{m}(\mathbf{q}) \exp \left[-\frac{t + Kq^2 + Lq^4 + \dots}{2V} |m(\mathbf{q})|^2 + \vec{h} \cdot \vec{m}(\mathbf{q}=0) \right]. \quad (\text{III.53})$$

The integral for $\mathbf{q} = \mathbf{0}$ is

$$Z_0 = V^{-n/2} \int_{-\infty}^{\infty} d\vec{m}(\mathbf{0}) \exp \left[-\frac{t}{2V} |m(\mathbf{0})|^2 + \vec{h} \cdot \vec{m}(\mathbf{0}) \right] = \left(\frac{2\pi}{t} \right)^{n/2} \exp \left[\frac{Vh^2}{2t} \right]. \quad (\text{III.54})$$

After performing the integrations for $\mathbf{q} \neq \mathbf{0}$, we obtain

$$Z = \exp \left[\frac{Vh^2}{2t} \right] \prod_{\mathbf{q}} \left(\frac{2\pi}{t + Kq^2 + Lq^4 + \dots} \right)^{n/2}. \quad (\text{III.55})$$

The total number of modes N , equals the number of original lattice points. Apart from a constant factor resulting from $(2\pi)^{nN/2}$, the free energy is

$$f(t, h) = -\frac{\ln Z}{V} = \frac{n}{2} \int_{B.Z.} \frac{d^d \mathbf{q}}{(2\pi)^d} \ln (t + Kq^2 + Lq^4 + \dots) - \frac{h^2}{2t}. \quad (\text{III.56})$$

The integral in eq.(III.56) is over the Brillouin zone, which for a hypercubic lattice of spacing a , is a cube of side $2\pi/a$ centered on the origin. However, we expect the singularities to originate from the long wavelength modes close to $\mathbf{q} = \mathbf{0}$. The contributions from the vicinity of the Brillouin zone edge are clearly analytic, since the logarithm can be simply expanded in powers of t for a finite q^2 . Thus to simplify the extraction of the singular behavior as $t \rightarrow 0$, we approximate the shape of the Brillouin zone by a hypersphere of radius $\Lambda \approx \pi/a$. The spherical symmetry of the integrand then allows us to write

$$f_{\text{sing.}}(t, h) = \frac{n}{2} K_d \int_0^\Lambda dq q^{d-1} \ln (t + Kq^2 + Lq^4 + \dots) - \frac{h^2}{2t}, \quad (\text{III.57})$$

where $K_d \equiv S_d/(2\pi)^d$, and S_d is the d -dimensional solid angle. The leading dependence of the integral on t can be extracted after rescaling q by a factor of $\sqrt{t/K}$, to

$$f_{\text{sing.}}(t, h) = \frac{n}{2} K_d \left(\frac{t}{K} \right)^{d/2} \int_0^{\Lambda\sqrt{K}/\sqrt{t}} dx x^{d-1} [\ln t + \ln (1 + x^2 + Ltx^4/K^2 + \dots)] - \frac{h^2}{2t}. \quad (\text{III.58})$$

Ignoring analytic contributions in t , the leading singular dependence of the free energy can be written as

$$f_{\text{sing.}}(t, h) = -t^{d/2} \left[A + \frac{h^2}{2t^{1+d/2}} \right] \equiv t^{2-\alpha} g_f(h/t^\Delta). \quad (\text{III.59})$$

Note that the higher order gradients, i.e. terms proportional to $L, \dots$, do not effect the singular behavior in eq.(III.59). On approaching the point $h = 0$ for $t = 0^+$, the singular part of the free energy is described by a homogeneous scaling form, with exponents

$$\alpha_+ = 2 - d/2 \quad , \quad \Delta = 1/2 + d/4. \quad (\text{III.60})$$

Since the ordered phase for $t < 0$ is not stable, the exponent β is undefined. The susceptibility $\chi \propto \partial^2 f / \partial h^2 \propto 1/t$, however, diverges with the exponent $\gamma_+ = 1$.

III.G The Gaussian Model (Renormalization Group)

The RG of the Gaussian model is most conveniently performed in terms of the Fourier modes. The goal is to evaluate the partition function

$$Z \sim \int \mathcal{D}\vec{m}(\mathbf{q}) \exp \left[- \int_0^\Lambda \frac{d^d \mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2 + Lq^4 + \dots}{2} \right) |m(\mathbf{q})|^2 + \vec{h} \cdot \vec{m}(\mathbf{0}) \right], \quad (\text{III.61})$$

indirectly via the three steps of RG. Note that the Brillouin zone is approximated by a hypersphere of radius Λ .

1. Coarse-Grain: Eliminating fluctuations at scales $a < x < ba$; is similar to removing Fourier modes with wavenumbers $\Lambda/b < q < \Lambda$. We thus break up the momenta into two subsets,

$$\{\vec{m}(\mathbf{q})\} = \{\vec{\sigma}(\mathbf{q}^>)\} \oplus \{\tilde{\vec{m}}(\mathbf{q}^<)\}, \quad (\text{III.62})$$

and write

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}^<) \int \mathcal{D}\vec{\sigma}(\mathbf{q}^>) e^{-\beta \mathcal{H}[\tilde{\vec{m}}, \vec{\sigma}]}. \quad (\text{III.63})$$

Since the two sets of modes are decoupled in the Gaussian model, the integration is trivial, and

$$Z \sim \exp \left[-\frac{n}{2} V \int_{\Lambda/b}^\Lambda \frac{d^d \mathbf{q}}{(2\pi)^d} \ln(t + Kq^2 + Lq^4 + \dots) \right] \times \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}^<) \exp \left[- \int_0^{\Lambda/b} \frac{d^d \mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2 + Lq^4 + \dots}{2} \right) |\tilde{m}(\mathbf{q})|^2 + \vec{h} \cdot \tilde{\vec{m}}(\mathbf{0}) \right]. \quad (\text{III.64})$$

2. Rescale: The partition function for the modes $\tilde{\vec{m}}(\mathbf{q}^<)$ is similar to the original, except that the upper cut-off has decreased to Λ/b , reflecting the coarsening in resolution. The rescaling $\mathbf{x}' = \mathbf{x}/b$ in real space, is equivalent to $\mathbf{q}' = b\mathbf{q}$ in *momentum space*, and restores the cut-off to its original value. The rescaled partition function is

$$Z = e^{-V\delta f_b(t)} \times \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}') \exp \left[- \int_0^\Lambda \frac{d^d \mathbf{q}'}{(2\pi)^d} b^{-d} \left(\frac{t + Kb^{-2}q'^2 + Lb^{-4}q'^4 + \dots}{2} \right) |\tilde{\vec{m}}(\mathbf{q}')|^2 + \vec{h} \cdot \tilde{\vec{m}}(\mathbf{0}) \right]. \quad (\text{III.65})$$

3. Renormalize: The final step of RG in real space is the renormalization of magnetization, $\vec{m}'(\mathbf{x}') = \tilde{\vec{m}}(\mathbf{x}')/\zeta$. Alternatively, we can renormalize the Fourier modes according to $\vec{m}'(\mathbf{q}') = \tilde{\vec{m}}(\mathbf{q}')/z$, resulting in

$$Z = e^{-V\delta f_b(t)} \times \int \mathcal{D}\vec{m}'(\mathbf{q}') \exp \left[- \int_0^\Lambda \frac{d^d \mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{t + Kb^{-2}q'^2 + Lb^{-4}q'^4 + \dots}{2} \right) |m'(\mathbf{q}')|^2 + z\vec{h} \cdot \vec{m}'(\mathbf{0}) \right]. \quad (\text{III.66})$$

(Note that the factors ζ and z , for rescalings of magnetization in real and Fourier space, are different.)

Eq.(III.66) indicates the renormalized modes are distributed according to a Gaussian Hamiltonian, with renormalized parameters,

$$\begin{cases} t' = z^2 b^{-d} t \\ h' = zh \\ K' = z^2 b^{-d-2} K \\ L' = z^2 b^{-d-4} L \\ \dots \end{cases} \quad (\text{III.67})$$

The singular point $t = h = 0$ is mapped onto itself as expected. To make the fluctuations scale invariant at this point, we must ensure that the remaining Hamiltonian, stays fixed. This is achieved by the choice of $z = b^{1+d/2}$, which sets $K' = K$, and makes the remaining parameters $L, \dots$ scale to zero. Away from criticality, the two relevant directions now scale as

$$\begin{cases} t' = b^2 t \\ h' = b^{1+d/2} h \end{cases}, \quad \implies \quad \begin{cases} y_t = 2 \\ y_h = 1 + d/2 \end{cases}. \quad (\text{III.68})$$

Using the results of sec.(III.D), we can identify the critical exponents,

$$\begin{aligned}\nu &= 1/y_t = 1/2, \\ \Delta &= \frac{y_h}{y_t} = \frac{1 + d/2}{2} = \frac{1}{2} + \frac{d}{4}, \\ \alpha &= 2 - d\nu = 2 - d/2,\end{aligned}$$

in agreement with the direct solution in the previous section.

The fixed point Hamiltonian ($t^* = h^* = L^* = \dots = 0$), in *real space* is

$$\beta\mathcal{H}^* = \frac{K}{2} \int_a d^d \mathbf{x} (\nabla m)^2. \quad (\text{III.69})$$

(The subscript a is placed on the integral as a reminder of the implicit short distance cut-off.) Under a simple rescaling $\mathbf{x} \mapsto \mathbf{x}'$, and $\vec{m}(\mathbf{x}) \mapsto \zeta \vec{m}'(\mathbf{x}')$; $K \mapsto K' = b^{d-2} \zeta^2 K$. Scale invariance is achieved with the choice $\zeta = b^{1-d/2}$. Forgetting the coarse-graining step, a general power of $\vec{m}(\mathbf{x})$, added as a small perturbation to $(\beta\mathcal{H})^*$, behaves as

$$\beta\mathcal{H}^* + u_n \int d^d \mathbf{x} m^n \quad \mapsto \quad \beta\mathcal{H}^* + u_n b^d \zeta^n \int d^d \mathbf{x}' m'^n, \quad (\text{III.70})$$

suggesting that such perturbations scale as

$$u'_n = b^d b^{n(\frac{2-d}{2})} u_n, \quad \implies \quad y_n = n - d \left(\frac{n}{2} - 1 \right). \quad (\text{III.71})$$

The values $y_1 = 1 + d/2$, and $y_2 = 2$, reproduce the exponents for y_h and y_t in eq.(III.68). The operators with higher powers are less relevant. The next most important operator in a system with spherical symmetry is $y_4 = 4 - d$, which is irrelevant for $d > 4$ but relevant for $d < 4$; $y_6 = 6 - 2d$ is relevant only for $d < 3$. Indeed the majority of operators are irrelevant at the Gaussian fixed point for $d > 2$.