II.C Spontaneous Symmetry Breaking and Goldstone Modes

For zero field, $\vec{h} = 0$, although the microscopic Hamiltonian has full rotational symmetry, the low–temperature phase does not. As a specific direction in n–space is selected for the net magnetization $\vec{M}$, there is a *spontaneously broken symmetry*, and a corresponding *long–range order* is established in the system. The original symmetry is still present globally, in the sense that if all local magnetizations $\vec{m}(\mathbf{x})$, are rotated together (i.e. if $\vec{m}(\mathbf{x}) \mapsto \Re\vec{m}(\mathbf{x})$), there is no change in energy. Such a rotation transforms one ordered state into an equivalent one. If a uniform rotation costs no energy, by continuity we expect a rotation that is slowly varying in space (e.g. $\vec{m}(\mathbf{x}) \mapsto \Re(\mathbf{x})\vec{m}(\mathbf{x})$, where $\Re(\mathbf{x})$ only has long wavelength variations) to cost very little energy. Such low energy excitations are called *Goldstone modes*. They are present in any system with a broken *continuous symmetry*. There are no Goldstone modes when a discrete symmetry is broken, since it is impossible to produce slowly varying rotations from one state to an equivalent one. Phonons are an example of Goldstone modes, corresponding to the breaking of translation and rotation symmetries by a crystal structure.

Let us explore the origin and consequences of Goldstone modes in the context of *superfluidity*. In analogy to Bose condensation, the superfluid phase has a macroscopic occupation of a single quantum ground state. The order parameter,

$$\psi(\mathbf{x}) \equiv \psi_{\Re} + i\psi_{\Im} \equiv |\psi(\mathbf{x})|e^{i\theta(\mathbf{x})}, \tag{II.11}$$

is the ground state component (overlap) of the actual wavefunction in the vicinity of $\mathbf{x}$. The phase of the wavefunction is not an observable quantity and should not appear in any physically measurable probability. For example, the effective coarse grained Hamiltonian can be obtained as an expansion,

$$\beta\mathcal{H} = \int d^d\mathbf{x}\left[\frac{K}{2}|\nabla\psi|^2 + \frac{t}{2}|\psi|^2 + u|\psi|^4 + \cdots\right]. \tag{II.12}$$

Clearly, eq.(II.12) is equivalent to the Landau–Ginzburg Hamiltonian with $n = 2$ ($\vec{m} \equiv (\psi_{\Re}, \psi_{\Im})$). The superfluid transition is signaled by the onset of a finite value of ψ for $t < 0$. Minimizing the Hamiltonian fixes the magnitude of ψ, but not its phase θ. Now consider a state with a slowly varying phase, i.e. with $\psi(\mathbf{x}) = \bar{\psi}e^{i\theta(\mathbf{x})}$. Inserting this form in the Hamiltonian yields an energy

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \frac{\bar{K}}{2}\int d^d\mathbf{x}(\nabla\theta)^2, \tag{II.13}$$

where $\bar{K} = K\bar{\psi}^2$. Taking advantage of translational symmetry, Eq.(II.13) can be decomposed into independent modes (in a region of volume V) by setting $\theta(\mathbf{x}) = \sum_q e^{i\mathbf{q}\cdot\mathbf{x}}\theta_{\mathbf{q}}/\sqrt{V}$, as

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \frac{\bar{K}}{2}\sum_{\mathbf{q}} q^2|\theta(\mathbf{q})|^2. \tag{II.14}$$

Clearly the long wavelength Goldstone modes cost little energy and are easily excited by thermal fluctuations.

Assuming that the amplitude of the order parameter is indeed uniform, the probability of a particular configuration is given by,

$$\mathcal{P}[\theta(\mathbf{x})] \propto \exp\left[-\frac{\bar{K}}{2}\int d^d\mathbf{x}(\nabla\theta)^2\right]. \tag{II.15}$$

Alternatively, in terms of the Fourier components,

$$\mathcal{P}[\theta(\mathbf{q})] \propto \exp\left[-\frac{\bar{K}}{2}\sum_{\mathbf{q}} q^2|\theta(\mathbf{q})|^2\right] \propto \prod_{\mathbf{q}} p(\theta_{\mathbf{q}}). \tag{II.16}$$

Each mode $\theta_{\mathbf{q}}$, is an *independent* random variable with a Gaussian distribution of zero mean, and with †

$$\langle\theta_{\mathbf{q}}\theta_{\mathbf{q}'}\rangle = \frac{\delta_{\mathbf{q},-\mathbf{q}'}}{\bar{K}q^2}. \tag{II.17}$$

† Note that the Fourier transform of a real field $\theta(\mathbf{x})$, is complex $\theta_{\mathbf{q}} = \theta_{\mathbf{q},\Re} + i\theta_{\mathbf{q},\Im}$. However, the number of fields is not doubled, due to the constraint of $\theta_{-\mathbf{q}} = \theta^*_{\mathbf{q}} = \theta_{\mathbf{q},\Re} - i\theta_{\mathbf{q},\Im}$. A Gaussian translational invariant weight has the generic form

$$\mathcal{P}\left[\{\theta_{\mathbf{q}}\}\right] \propto \prod_{\mathbf{q}} \exp\left[-\frac{K(q)}{2}\theta_{\mathbf{q}}\theta_{-\mathbf{q}}\right] = \prod_{\mathbf{q}>0} \exp\left[-\frac{2K(q)}{2}\left(\theta^2_{\mathbf{q},\Re} + \theta^2_{\mathbf{q},\Im}\right)\right].$$

While the first product is over all $\mathbf{q}$, the second is restricted to half of the space. There are clearly no cross correlations for differing $\mathbf{q}$, and the Gaussian variances are

$$\left\langle\theta^2_{\mathbf{q},\Re}\right\rangle = \left\langle\theta^2_{\mathbf{q},\Im}\right\rangle = \frac{1}{2K(q)},$$

from which we can immediately construct

$$\langle\theta_{\mathbf{q}}\theta_{\mp\mathbf{q}}\rangle = \left\langle\theta^2_{\mathbf{q},\Re}\right\rangle \pm \left\langle\theta^2_{\mathbf{q},\Im}\right\rangle = \frac{1\pm 1}{2K(q)}.$$

From eq.(II.17) we can calculate the correlations in the phase $\theta(\mathbf{x})$ in real space. Clearly $\langle\theta(\mathbf{x})\rangle = 0$ by symmetry, while

$$\langle\theta(\mathbf{x})\theta(\mathbf{x}')\rangle = \frac{1}{V}\sum_{\mathbf{q},\mathbf{q}'} e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'}\langle\theta_{\mathbf{q}}\theta_{\mathbf{q}'}\rangle = \frac{1}{V}\sum_{\mathbf{q}}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{\bar{K}q^2}. \tag{II.18}$$

In the continuum limit, the sum can be replaced by an integral ($\sum_{\mathbf{q}} \mapsto V\int d^d\mathbf{q}/(2\pi)^d$), and

$$\langle\theta(\mathbf{x})\theta(\mathbf{x}')\rangle = \int\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{\bar{K}q^2} = -\frac{C_d(\mathbf{x}-\mathbf{x}')}{\bar{K}}. \tag{II.19}$$

The function,

$$C_d(\mathbf{x}) = -\int\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{q^2}, \tag{II.20}$$

is the Coulomb potential due to a unit charge at the origin in a d–dimensional space, since it is the solution to

$$\nabla^2 C_d(\mathbf{x}) = \int\frac{d^d\mathbf{q}}{(2\pi)^d}\,\frac{q^2}{q^2}\,e^{i\mathbf{q}\cdot\mathbf{x}} = \delta^d(\mathbf{x}). \tag{II.21}$$

We can easily find a solution by using Gauss' theorem,

$$\int d^d x\nabla^2 C_d = \oint dS\cdot\nabla C_d \quad .$$

For a spherically symmetric solution, $\nabla C_d = (dC_d/dx)\hat{x}$, and the above equation simplifies to

$$1 = S_d x^{d-1}\frac{dC_d}{dx}, \tag{II.22}$$

where

$$S_d = \frac{2\pi^{d/2}}{(d/2-1)!}, \tag{II.23}$$

is the total solid angle (area of unit sphere) in d dimensions. Hence

$$\frac{dC_d}{dx} = \frac{1}{S_d x^{d-1}}, \quad\Longrightarrow\quad C_d(x) = \frac{x^{2-d}}{(2-d)S_d} + c_0, \tag{II.24}$$

where c_0 is a constant of integration.

The long distance behavior of $C_d(x)$ changes dramatically at $d = 2$, as

$$\lim_{x\to\infty} C_d(x) = \begin{cases} c_0 & d > 2 \\ \dfrac{x^{2-d}}{(2-d)S_d} & d < 2 \\ \dfrac{\ln(x)}{2\pi} & d = 2 \end{cases} \quad . \tag{II.25}$$

The constant of integration can obtained by looking at

$$\langle[\theta(\mathbf{x})-\theta(\mathbf{x}')]^2\rangle = 2\langle\theta(\mathbf{x})^2\rangle - 2\langle\theta(\mathbf{x})\theta(\mathbf{x}')\rangle, \tag{II.26}$$

which goes to zero as $\mathbf{x}\to\mathbf{x}'$. Hence,

$$\langle[\theta(\mathbf{x})-\theta(\mathbf{x}')]^2\rangle = \frac{2\left(|\mathbf{x}-\mathbf{x}'|^{2-d}-a^{2-d}\right)}{\bar{K}(2-d)S_d}, \tag{II.27}$$

where a is of the order of the lattice spacing.

For $d>2$, the phase fluctuations are finite, while they become asymptotically large for $d\leq 2$. Since the phase is bounded by 2π, this implies that long range order in the phase is destroyed. This result becomes more apparent by examining the effect of phase fluctuations on the two point correlation function

$$\langle\psi(\mathbf{x})\psi^*(\mathbf{0})\rangle = \bar{\psi}^2\langle e^{i[\theta(\mathbf{x})-\theta(\mathbf{0})]}\rangle. \tag{II.28}$$

(Since amplitude fluctuations are ignored, we are in fact looking at a *transverse* correlation function.) We shall prove later on that for any collection of Gaussian distributed variables,

$$\langle\exp(\alpha\theta)\rangle = \exp\left(\frac{\alpha^2}{2}\langle\theta^2\rangle\right).$$

Taking this result for granted, we obtain

$$\langle\psi(\mathbf{x})\psi^*(0)\rangle = \bar{\psi}^2\exp\left[-\frac{1}{2}\langle[\theta(\mathbf{x})-\theta(\mathbf{0})]^2\rangle\right] = \bar{\psi}^2\exp\left[-\frac{x^{2-d}-a^{2-d}}{\bar{K}(2-d)S_d}\right], \tag{II.29}$$

and asymptotically

$$\lim_{x\to\infty}\langle\psi(\mathbf{x})\psi^*(\mathbf{0})\rangle = \begin{cases}\bar{\psi'}^2 & \text{for } d>2\\ 0 & \text{for } d\leq 2\end{cases}. \tag{II.30}$$

The saddle point approximation to the order parameter $\bar{\psi}$, was obtained by ignoring fluctuations. The above result indicates that inclusion of phase fluctuations leads to a reduction of order in $d>2$, and its complete destruction in $d\leq 2$.

The above example typifies a more general result known as the *Mermin–Wagner theorem*. The theorem states that there is no spontaneous breaking of a continuous symmetry in systems with short range interactions in dimensions $d\leq 2$. Some corollaries to this theorem are:

(1) The borderline dimensionality of two, known as the *lower critical dimension*, has to be treated carefully. As we shall demonstrate later on in the course, there is in fact a phase transition for the two dimensional superfluid, although there is no true long range order.

(2) There are no Goldstone modes when the broken symmetry is discrete (e.g. for $n=1$). In such cases, long range order is possible down to the lower critical dimension of $d_\ell=1$.