VIII. Dissipative Dynamics

VIII.A Brownian Motion of a Particle

Observations under a microscope indicate that a dust particle in a liquid drop undergoes a random jittery motion. This is because of the random impacts of the much smaller fluid particles. The theory of such (*Brownian*) motion was developed by Einstein in 1905 and starts with the equation of motion for the particle. The displacement $\vec{x}(t)$, of a particle of mass m is governed by,

$$m\,\ddot{\vec{x}} = -\frac{\dot{\vec{x}}}{\mu} - \frac{\partial \mathcal{V}}{\partial \vec{x}} + \vec{f}_{\text{random}}(t). \tag{VIII.1}$$

The three forces acting on the particle are:

(i) A friction force due to the viscosity of the fluid. For a spherical particle of radius R, the mobility in the low Reynolds number limit is given by $\mu = (6\pi\bar{\eta}R)^{-1}$, where $\bar{\eta}$ is the specific viscosity.

(ii) The force due to the external potential $\mathcal{V}(\vec{x})$, e.g. gravity.

(iii) A random force of zero mean due to the impacts of fluid particles.

The viscous term usually dominates the inertial one (i.e. the motion is overdamped), and we shall henceforth ignore the acceleration term. Eq.(VIII.1) now reduces to a *Langevin equation*,

$$\dot{\vec{x}} = \vec{v}(\vec{x}) + \vec{\eta}(t), \tag{VIII.2}$$

where $\vec{v}(\vec{x}) = -\mu\partial\mathcal{V}/\partial\vec{x}$ is the *deterministic* velocity. The *stochastic* velocity, $\vec{\eta}(t) = \mu\vec{f}_{\text{random}}(t)$, has zero mean,

$$\langle \vec{\eta}(t) \rangle = 0. \tag{VIII.3}$$

It is usually assumed that the probability distribution for the noise in velocity is Gaussian, i.e.

$$\mathcal{P}\left[\vec{\eta}(t)\right] \propto \exp\left[-\int d\tau\,\frac{\eta(\tau)^2}{4D}\right]. \tag{VIII.4}$$

Note that different components of the noise, and at different times, are independent, and the covariance is

$$\langle \eta_\alpha(t)\eta_\beta(t')\rangle = 2D\delta_{\alpha,\beta}\delta(t-t'). \tag{VIII.5}$$

The parameter D is related to *diffusion* of particles in the fluid. In the absence of any potential, $\mathcal{V}(\vec{x}) = 0$, the position of a particle at time t is given by

$$\vec{x}(t) = \vec{x}(0) + \int_0^t d\tau \vec{\eta}(\tau).$$

Clearly the separation $\vec{x}(t) - \vec{x}(0)$ which is the sum of random Gaussian variables is itself Gaussian distributed with mean zero, and a varaince

$$\left\langle (\vec{x}(t) - \vec{x}(0))^2 \right\rangle = \int_0^t d\tau_1 d\tau_2 \left\langle \vec{\eta}(\tau_1) \cdot \vec{\eta}(\tau_2) \right\rangle = 3 \times 2Dt.$$

For an ensemble of particles released at $\vec{x}(t) = 0$, i.e. with $\mathcal{P}(\vec{x}, t=0) = \delta^3(\vec{x})$, the particles at time t are distributed according to

$$\mathcal{P}(\vec{x}, t) = \left(\frac{1}{\sqrt{4\pi Dt}} \right)^{3/2} \exp\left[-\frac{x^2}{4Dt} \right],$$

which is the solution to the diffusion equation

$$\frac{\partial \mathcal{P}}{\partial t} = D \nabla^2 \mathcal{P}.$$

A simple example is provided by a particle connected to a Hookian spring, with $\mathcal{V}(\vec{x}) = Kx^2/2$. The deterministic velocity is now $\vec{v}(\vec{x}) = -\mu K \vec{x}$, and the Langevin equation, $\dot{\vec{x}} = -\mu K \vec{x} + \vec{\eta}(t)$, can be rearranged as

$$\frac{d}{dt}\left[e^{\mu K t} \vec{x}(t) \right] = e^{\mu K t} \vec{\eta}(t). \tag{VIII.6}$$

Integrating the equation from 0 to t yields

$$e^{\mu K t} \vec{x}(t) - \vec{x}(0) = \int_0^t d\tau e^{\mu K \tau} \vec{\eta}(\tau), \tag{VIII.7}$$

and

$$\vec{x}(t) = \vec{x}(0) e^{-\mu K t} + \int_0^t d\tau e^{-\mu K (t-\tau)} \vec{\eta}(\tau). \tag{VIII.8}$$

Averaging over the noise indicates that the mean position,

$$\langle \vec{x}(t) \rangle = \vec{x}(0) e^{-\mu K t}, \tag{VIII.9}$$

decays with a characteristic *relaxation time*, $\tau = 1/(\mu K)$. Fluctuations around the mean behave as

$$
\begin{aligned}
\left\langle \left(\vec{x}(t) - \langle \vec{x}(t) \rangle \right)^2 \right\rangle =& \int_0^t d\tau_1 d\tau_2 e^{-\mu K (2t-\tau_1-\tau_2)} \overbrace{\langle \vec{\eta}(\tau_1) \cdot \vec{\eta}(\tau_2) \rangle}^{2D\delta(\tau_1-\tau_2)\times 3} \\
=& 6D \int_0^t d\tau e^{-2\mu K (t-\tau)} \\
=& \frac{3D}{\mu K} \left[1 - e^{-2\mu K t} \right] \quad \stackrel{t\to\infty}{\longrightarrow} \quad \frac{3D}{\mu K}.
\end{aligned}
\tag{VIII.10}
$$

However, once the dust particle reaches equilibrium with the fluid at a temperature T, its probability distribution must satisfy the normalized Boltzmann weight

$$
\mathcal{P}_{\text{eq.}}(\vec{x}) = \left(\frac{K}{2\pi k_B T} \right)^{3/2} \exp\left[-\frac{K x^2}{2 k_B T} \right], \tag{VIII.11}
$$

yielding $\left\langle x^2 \right\rangle = 3 k_B T / K$. Since the dynamics is expected to bring the particle to equilibrium with the fluid at temperature T, eq.(VIII.10) implies the condition

$$
D = k_B T \mu\,. \tag{VIII.12}
$$

This is the Einstein relation connecting the *fluctuations* of noise to the *dissipation* in the medium.

Clearly the Langevin equation at long times reproduces the correct mean and variance for a particle in equilibrium at a temperature T in the potential $\mathcal{V}(\vec{x}) = Kx^2/2$, provided that eq.(VIII.12) is satisfied. Can we show that the whole probability distribution evolves to the Boltzmann weight for any potential? Let $\mathcal{P}(\vec{x}, t) \equiv \langle \vec{x} | \mathcal{P}(t) | 0 \rangle$ denote the probability density of finding the particle at $\vec{x}$ at time t, given that it was at 0 at $t = 0$. This probability can be constructed recursively by noting that a particle found at $\vec{x}$ at time $t + \epsilon$ must have arrived from some other point $\vec{x}\,'$ at t. Adding up all such probabilities yields

$$
\mathcal{P}(\vec{x}, t+\epsilon) = \int d^3\vec{x}\,' \, \mathcal{P}(\vec{x}\,', t) \, \langle \vec{x} | T_\epsilon | \vec{x}\,' \rangle, \tag{VIII.13}
$$

where $\langle \vec{x} | T_\epsilon | \vec{x}\,' \rangle \equiv \langle \vec{x} | \mathcal{P}(\epsilon) | \vec{x}\,' \rangle$ is the transition probability. For $\epsilon \ll 1$,

$$
\vec{x} = \vec{x}\,' + \vec{v}\,(\vec{x}\,')\epsilon + \vec{\eta}_\epsilon \quad , \tag{VIII.14}
$$

where $\vec{\eta}_\epsilon = \int_t^{t+\epsilon} d\tau \vec{\eta}(\tau)$. Clearly, $\langle \vec{\eta}_\epsilon \rangle = 0$, and $\langle \eta_\epsilon^2 \rangle = 2D\epsilon \times 3$, and following eq.(VIII.4),

$$p(\vec{\eta}_\epsilon) = \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp\left[-\frac{\eta_\epsilon^2}{4D\epsilon} \right]. \tag{VIII.15}$$

The transition rate is simply the probability of finding a noise of the right magnitude according to eq.(VIII.14), and

$$\begin{aligned} \langle \vec{x} | T(\epsilon) | \vec{x}\,' \rangle = p(\eta_\epsilon) &= \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp\left[-\frac{\left(\vec{x} - \vec{x}\,' - \epsilon \vec{v}(\vec{x}\,') \right)^2}{4D\epsilon} \right] \\ &= \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp\left[-\epsilon \frac{\left(\dot{\vec{x}} - \vec{v}(\vec{x}) \right)^2}{4D} \right]. \end{aligned} \tag{VIII.16}$$

By subdividing the time interval t, into infinitesimal segments of size ϵ, repeated application of the above evolution operator yields

$$\begin{aligned} \mathcal{P}(\vec{x}, t) &= \left\langle \vec{x} \left| T(\epsilon)^{t/\epsilon} \right| 0 \right\rangle \\ &= \int_{(0,0)}^{(\vec{x},t)} \frac{\mathcal{D}\vec{x}(\tau)}{\mathcal{N}} \exp\left[-\int_0^t d\tau \frac{\left(\dot{\vec{x}} - \vec{v}(\vec{x}) \right)^2}{4D} \right]. \end{aligned} \tag{VIII.17}$$

The integral is over all paths connecting the initial and final points; each path's weight is related to its deviation from the classical trajectory, $\dot{\vec{x}} = \vec{v}(\vec{x})$. The recursion relation (eq.(VIII.13)),

$$\mathcal{P}(\vec{x}, t) = \int d^3\vec{x}\,' \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp\left[-\frac{\left(\vec{x} - \vec{x}\,' - \epsilon \vec{v}(\vec{x}\,') \right)^2}{4D\epsilon} \right] \mathcal{P}(\vec{x}\,', t - \epsilon), \tag{VIII.18}$$

can be simplified by the change of variables,

$$\begin{aligned} \vec{y} &= \vec{x}\,' + \epsilon \vec{v}(\vec{x}\,') - \vec{x} \quad \Longrightarrow \\ d^3\vec{y} &= d^3\vec{x}\,' \left(1 + \epsilon \nabla \cdot \vec{v}(\vec{x}') \right) = d^3\vec{x}\,' \left(1 + \epsilon \nabla \cdot \vec{v}(\vec{x}) + \mathcal{O}(\epsilon^2) \right). \end{aligned} \tag{VIII.19}$$

Keeping only terms at order of ϵ, we obtain

$$\begin{aligned} \mathcal{P}(\vec{x}, t) &= \left[1 - \epsilon \nabla \cdot \vec{v}(\vec{x}) \right] \int d^3\vec{y} \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} e^{-\frac{y^2}{4D\epsilon}} \, \mathcal{P}(\vec{x} + \vec{y} - \epsilon \vec{v}(\vec{x}), t - \epsilon) \\ &= \left[1 - \epsilon \nabla \cdot \vec{v}(\vec{x}) \right] \int d^3\vec{y} \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} e^{-\frac{y^2}{4D\epsilon}} \times \\ &\quad \left[\mathcal{P}(\vec{x}, t) + (\vec{y} - \epsilon \vec{v}(\vec{x})) \cdot \nabla \mathcal{P} + \frac{y_i y_j - 2\epsilon y_i v_j + \epsilon^2 v_i v_j}{2} \nabla_i \nabla_j \mathcal{P} - \epsilon \frac{\partial \mathcal{P}}{\partial t} + \mathcal{O}(\epsilon^2) \right] \\ &= \left[1 - \epsilon \nabla \cdot \vec{v}(\vec{x}) \right] \left[\mathcal{P} - \epsilon \vec{v} \cdot \nabla + \epsilon D \nabla^2 \mathcal{P} - \epsilon \frac{\partial \mathcal{P}}{\partial t} + \mathcal{O}(\epsilon^2) \right]. \end{aligned} \tag{VIII.20}$$

Equating terms at order of ϵ leads to the *Fokker-Planck equation*,

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot \vec{J} = 0, \qquad \text{with} \qquad \vec{J} = \vec{v}\mathcal{P} - D\nabla\mathcal{P} \quad . \tag{VIII.21}$$

The Fokker-Planck equation is simply the statement of conservation of probability. The probability current has a deterministic component $\vec{v}\mathcal{P}$, and a stochastic part $-D\nabla\mathcal{P}$. A *stationary distribution*, $\partial\mathcal{P}/\partial t = 0$, is obtained if the net current vanishes. It is now easy to check that the Boltzmann weight, $\mathcal{P}_{\text{eq.}}(\vec{x}) \propto \exp[-\mathcal{V}(\vec{x})/k_B T]$, with $\nabla\mathcal{P}_{\text{eq.}} = \vec{v}\mathcal{P}_{\text{eq.}}/(\mu k_B T)$, leads to a stationary state as long as the fluctuation–dissipation condition in eq.(VIII.12) is satisfied.

VIII.B Equilibrium Dynamics of a Field

The next step is to generalize the Langevin formalism to a collection of degrees of freedom, most conveniently described by a continuous field. Let us consider the order parameter field $\vec{m}(\mathbf{x}, t)$ of a magnet. In equilibrium, the probability to find a coarse-grained configuration of the magnetization field is governed by the Boltzmann weight of the Landau-Ginzburg Hamiltonian

$$\mathcal{H}[\vec{m}] = \int d^d\mathbf{x} \left[\frac{r}{2}m^2 + um^4 + \frac{K}{2}(\nabla m)^2 + \cdots \right]. \tag{VIII.22}$$

(To avoid confusion with time, the coefficient of the quadratic term is changed from t to r.) Clearly the above energy functional contains no kinetic terms, and should be regarded as the analog of the potential energy $\mathcal{V}(\vec{x})$ employed in the previous section. To construct a Langevin equation governing the dynamics of the field $\vec{m}(\mathbf{x})$, we first calculate the analogous *force* on each field element from the variations of this potential energy. The *functional derivative* of eq.(VIII.22) yields

$$F_i(\mathbf{x}) = -\frac{\delta\mathcal{H}[\vec{m}]}{\delta m_i(\mathbf{x})} = -rm_i - 4um_i|\vec{m}|^2 + K\nabla^2 m_i. \tag{VIII.23}$$

The straightforward analog of eq.(VIII.2) is

$$\frac{\partial m_i(\mathbf{x}, t)}{\partial t} = \mu F_i(\mathbf{x}) + \eta_i(\mathbf{x}, t), \tag{VIII.24}$$

with a random velocity, $\vec{\eta}$, such that

$$\langle \eta_i(\mathbf{x}, t) \rangle = 0, \qquad \text{and} \qquad \langle \eta_i(\mathbf{x}, t)\eta_j(\mathbf{x}', t') \rangle = 2D\delta_{ij}\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{VIII.25}$$

The resulting Langevin equation,

$$\frac{\partial \vec{m}(\mathbf{x},t)}{\partial t} = -\mu r \vec{m} - 4\mu u m^2 \vec{m} + \mu K \nabla^2 \vec{m} + \vec{\eta}(\mathbf{x},t), \tag{VIII.26}$$

is known as the *time dependent Landau-Ginzburg equation.* Because of the nonlinear term $m^2\vec{m}$, it is not possible to integrate this equation exactly. To gain some insight into its behavior we start with the disordered phase of the model which is well described by the *Gaussian weight* with $u = 0$. The resulting linear equation is then easily solved by examining the Fourier components,

$$\vec{m}(\mathbf{q},t) = \int d^d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \vec{m}(\mathbf{x},t), \tag{VIII.27}$$

which evolve according to

$$\frac{\partial \vec{m}(\mathbf{q},t)}{\partial t} = -\mu(r + Kq^2)\ \vec{m}(\mathbf{q},t) + \vec{\eta}(\mathbf{q},t). \tag{VIII.28}$$

The Fourier transformed noise,

$$\vec{\eta}(\mathbf{q},t) = \int d^d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \vec{\eta}(\mathbf{x},t), \tag{VIII.29}$$

has zero mean, $\langle \eta_i(\mathbf{q},t)\rangle = 0$, and correlations

$$\begin{aligned}\langle \eta_i(\mathbf{q},t)\eta_j(\mathbf{q}',t')\rangle = &\int d^d\mathbf{x} d^d\mathbf{x}'\ e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'}\ \overbrace{\langle \eta_i(\mathbf{x},t)\eta_j(\mathbf{x}',t')\rangle}^{2D\delta_{ij}\delta^d(\mathbf{x}-\mathbf{x}')\ \delta(t-t')} \\ =&2D\delta_{ij}\delta(t-t') \int d^d\mathbf{x}\, e^{i\mathbf{x}\cdot(\mathbf{q}+\mathbf{q}')} \\ =&2D\delta_{ij}\delta(t-t')(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}').\end{aligned} \tag{VIII.30}$$

Each Fourier mode in eq.(VIII.28) now behaves as an independent particle connected to a spring as in eq.(VIII.6). Introducing a decay rate

$$\gamma(\mathbf{q}) \equiv \frac{1}{\tau(\mathbf{q})} = \mu(r + Kq^2), \tag{VIII.31}$$

the evolution of each mode is similar to eq.(VIII.8), and follows

$$\vec{m}(\mathbf{q},t) = \vec{m}(\mathbf{q},0)e^{-\gamma(\mathbf{q})t} + \int_0^t d\tau\ e^{-\gamma(\mathbf{q})(t-\tau)} \vec{\eta}(\mathbf{q},t). \tag{VIII.32}$$

Fluctuations in each mode decay with a different *relaxation time* $\tau(\mathbf{q})$; $\langle \vec{m}(\mathbf{q},t)\rangle = \vec{m}(\mathbf{q},0)\exp[-t/\tau(\mathbf{q})]$. When in equilibrium, the order parameter in the Gaussain model is correlated over the length scale $\xi = \sqrt{K/r}$. In considering relaxation to equilibrium, we find that at length scales larger than xi (or $q \ll 1/\xi$), the relaxation time saturates $\tau_{\max} = 1/(\mu r)$. On approaching the singular point of the Gaussian model at $r = 0$, the time required to reach equilibrium diverges. This phenomena is know as *critical slowing down*, and is also present for the non-linear equation, albeit with modified exponents. The critical point is thus characterized by diverging *length* and *time* scales. For the critical fluctuations at distances shorter than the correlation length ξ, the characteristic time scale grows with wavelength as $\tau(q) \approx (\mu K q^2)^{-1}$. The scaling relation between the critical length and time scales is described by a *dynamic exponent* z, as $\tau \propto \lambda^z$. The value of $z = 2$ for the critical Gaussian model is reminiscent of diffusion processes.

Time dependent correlation functions are obtained from

$$\begin{aligned}
\langle m_i(\mathbf{q},t)m_j(\mathbf{q}',t)\rangle_c =& \int_0^t d\tau_1 d\tau_2 e^{-\gamma(\mathbf{q})(t-\tau_1)-\gamma(\mathbf{q}')(t-\tau_2)} \overbrace{\langle \eta_i(\mathbf{q},\tau_1)\eta_j(\mathbf{q}',\tau_2)\rangle}^{2D\delta_{ij}\delta(\tau_1-\tau_2)(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')} \\
=&(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\, 2D\delta_{ij} \int_0^t d\tau e^{-2\gamma(\mathbf{q})(t-\tau)} \\
=&(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\delta_{ij}\, \frac{D}{\gamma(\mathbf{q})}\left(1-e^{-2\gamma(\mathbf{q})t}\right) \\
\stackrel{t\to\infty}{\longrightarrow}& \quad (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\delta_{ij}\frac{D}{\mu(r+Kq^2)}.
\end{aligned} \tag{VIII.33}$$

However, direct diagonalization of the Hamiltonian in eq.(VIII.22) with $u = 0$ gives

$$\mathcal{H} = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{(r+Kq^2)}{2}|\vec{m}(\mathbf{q})|^2, \tag{VIII.34}$$

leading to the equilibrium correlation functions

$$\langle m_i(\mathbf{q})m_j(\mathbf{q}')\rangle = (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')\delta_{ij}\,\frac{k_BT}{r+Kq^2}. \tag{VIII.35}$$

Comparing equations (VIII.33) and (VIII.35) indicates that the long-time dynamics reproduce the correct equilibrium behavior if the fluctuation–dissipation condition, $D = k_BT\mu$, is satisfied.

In fact, quite generally, the single particle Fokker-Planck equation (VIII.21) can be generalized to describe the evolution of the whole probability functional, $\mathcal{P}([\vec{m}(\mathbf{x})], t)$, as

$$\frac{\partial \mathcal{P}([\vec{m}(\mathbf{x})], t)}{\partial t} = -\int d^d\mathbf{x} \frac{\delta}{\delta m_i(\mathbf{x})}\left[-\mu \frac{\delta \mathcal{H}}{\delta m_i(\mathbf{x})}\mathcal{P} - D\frac{\delta \mathcal{P}}{\delta m_i(\mathbf{x})}\right]. \tag{VIII.36}$$

For the equilibrium Boltzmann weight

$$\mathcal{P}_{\text{eq.}}[\vec{m}(\mathbf{x})] \propto \exp\left[-\frac{\mathcal{H}[\vec{m}(\mathbf{x})]}{k_B T}\right], \tag{VIII.37}$$

the functional derivative results in

$$\frac{\delta \mathcal{P}_{\text{eq.}}}{\delta m_i(\mathbf{x})} = -\frac{1}{k_B T}\frac{\delta \mathcal{H}}{\delta m_i(\mathbf{x})}\mathcal{P}_{\text{eq.}}. \tag{VIII.38}$$

The total probability current,

$$J[h(\mathbf{x})] = \left[-\mu \frac{\delta \mathcal{H}}{\delta m_i(\mathbf{x})} + \frac{D}{k_B T}\frac{\delta \mathcal{H}}{\delta m_i(\mathbf{x})}\right]\mathcal{P}_{\text{eq.}}, \tag{VIII.39}$$

vanishes if the fluctuation–dissipation condition, $D = \mu k_B T$, is satisfied. Once again, the Einstein equation ensures that the equilibrium weight indeed describes a steady state.

VIII.C Dynamics of a Conserved field

In fact it is possible to obtain the correct equilibrium weight with $\mathbf{q}$ dependent mobility and noise, as long as the generalized fluctuation–dissipation condition,

$$D(\mathbf{q}) = k_B T \mu(\mathbf{q}), \tag{VIII.40}$$

holds. This generalized condition is useful in considering the dissipative dynamics of a *conserved* field. The prescription that leads to the Langevin equations (VIII.23)–(VIII.25), does not conserve the field in the sense that $\int d^d\mathbf{x}\vec{m}(\mathbf{x}, t)$ can change with time. (Although this quantity is on average zero for $r > 0$, it undergoes stochastic fluctuations.) If we are dealing with the a binary mixture ($n = 1$), the order parameter which measures the difference between densities of the two components is conserved. Any concentration that is removed from some part of the system must go to a neighboring region in any realistic dynamics. Let us then consider a dynamical process constrained such that

$$\frac{d}{dt}\int d^d\mathbf{x}\, \vec{m}(\mathbf{x}, t) = \int d^d\mathbf{x}\, \frac{\partial \vec{m}(\mathbf{x}, t)}{\partial t} = \vec{0}. \tag{VIII.41}$$

How can we construct a dynamical equation that satisfies eq.(VIII.41)? The integral clearly vanishes if the integrand is a total divergence, i.e.

$$\frac{\partial m_i(\mathbf{x},t)}{\partial t} = -\nabla \cdot \mathbf{j}_i + \eta_i(\mathbf{x},t). \tag{VIII.42}$$

The noise itself must be a total divergence, $\eta_i = -\nabla \cdot \sigma_i$, and hence in Fourier space,

$$\langle \eta_i(\mathbf{q},t)\rangle = 0, \quad \text{and} \quad \langle \eta_i(\mathbf{q},t)\eta_j(\mathbf{q}',t')\rangle = 2D\delta_{ij}q^2\delta(t-t')(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}'). \tag{VIII.43}$$

We can now take advantage of the generalized Einstein relation in eq.(VIII.40) to ensure the correct equilibrium distribution by setting,

$$\mathbf{j}_i = \mu\nabla \cdot \left(-\frac{\delta \mathcal{H}}{\delta m_i(\mathbf{x})}\right). \tag{VIII.44}$$

The standard terminology for such dynamical equations is provided by Hohenberg and Halperin: In **model A** dynamics the field $\vec{m}$ is *not conserved*, and the mobility and diffusion coefficients are constants. In **model B** dynamics the field $\vec{m}$ is *conserved*, and $\hat{\mu} = -\mu\nabla^2$ and $\hat{D} = -D\nabla^2$.

Let us now reconsider the Gaussian model ($u = 0$), this time with a conserved order parameter, with model B dynamics

$$\frac{\partial \vec{m}(\mathbf{x},t)}{\partial t} = \mu r \nabla^2 \vec{m} - \mu K \nabla^4 \vec{m} + \vec{\eta}(\mathbf{x},t). \tag{VIII.45}$$

The evolution of each Fourier mode is given by

$$\frac{\partial \vec{m}(\mathbf{q},t)}{\partial t} = -\mu q^2(r + Kq^2)\vec{m}(\mathbf{q},t) + \vec{\eta}(\mathbf{q},t) \equiv -\frac{\vec{m}(\mathbf{q},t)}{\tau(\mathbf{q})} + \vec{\eta}(\mathbf{q},t). \tag{VIII.46}$$

Because of the constraints imposed by the conservation law, the relaxation of the field is more difficult, and slower. The relaxation times diverge even away from criticality. Depending on wavelength, we find scaling between length and time scales with dynamic exponents z, according to

$$\tau(\mathbf{q}) = \frac{1}{\mu q^2(r + Kq^2)} \approx \begin{cases} q^{-2} & \text{for } q \ll \xi^{-1} \qquad (z=2) \\ q^{-4} & \text{for } q \gg \xi^{-1} \qquad (z=4) \end{cases}. \tag{VIII.47}$$

The equilibrium behavior is unchanged, and

$$\lim_{t\to\infty} \left\langle |\vec{m}(\mathbf{q},t)|^2 \right\rangle = n\frac{Dq^2}{\mu q^2(r+Kq^2)} = \frac{nD}{\mu(r+Kq^2)}, \tag{VIII.48}$$

as before. Thus the same static behavior can be achieved by different dynamics. The static exponents (e.g. ν) are determined by the equilibrium (stationary) state and are unchanged, while the dynamic exponents may be different. As a result, dynamical critical phenomena involve many more universality classes than the corresponding static ones.