## VII.C Perturbative Computation of Dielectric Response

The two partition functions in eq.(VII.58) are independent and can be calculated separately. As the Gaussian partition function is analytic, any phase transitions of the XY model must originate in the Coulomb gas. As briefly discussed earlier, in the low temperature phase the charges appear only in the small density of tightly bound dipole pairs. The dipoles dissociate in the high temperature phase, forming a plasma. The two phases can be distinguished by examining the interaction between two *external* test charges at a large separation $X$. In the absence of any *internal* charges (for $y_0 = 0$) in the medium, the two particles interact by the *bare* Coulomb interaction $C(X)$. A finite density of internal charges for small $y_0$ partially screens the external charges, and reduces the interaction between the test charges to $C(X)/\varepsilon$, where $\varepsilon$ is an effective *dielectric* constant. There is an *insulator to metal* transition at sufficiently large $y_0$. In the metallic (plasma) phase, the external charges are completely screened and their effective interaction decays exponentially.

To quantify the above picture, we shall compute the effective interaction between two *external* charges at $\mathbf{x}$ and $\mathbf{x}'$, perturbatively in the fugacity $y_0$. To lowest order, we need to include configurations with two *internal* charges (at $\mathbf{y}$ and $\mathbf{y}'$), and

$$\begin{aligned}
&e^{-\beta\mathcal{V}(\mathbf{x}-\mathbf{x}')} = e^{-4\pi^2KC(\mathbf{x}-\mathbf{x}')}\times \\
&\frac{\left[1+y_0^2\int d^2\mathbf{y}d^2\mathbf{y}'\, e^{-4\pi^2KC(\mathbf{y}-\mathbf{y}')+4\pi^2K\left[C(\mathbf{x}-\mathbf{y})-C(\mathbf{x}-\mathbf{y}')-C(\mathbf{y}'-\mathbf{x})+C(\mathbf{x}'-\mathbf{y}')\right]}+\mathcal{O}(y_0^4)\right]}{\left[1+y_0^2\int d^2\mathbf{y}d^2\mathbf{y}'\, e^{-4\pi^2KC(\mathbf{y}-\mathbf{y}')}+\mathcal{O}(y_0^4)\right]} \\
&= e^{-4\pi^2KC(\mathbf{x}-\mathbf{x}')}\left[1+y_0^2\int d^2\mathbf{y}d^2\mathbf{y}'\, e^{-4\pi^2KC(\mathbf{y}-\mathbf{y}')}\left(e^{4\pi^2KD(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')}-1\right)+\mathcal{O}(y_0^4)\right],
\end{aligned} \tag{VII.60}$$

where $D(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')$ is the interaction *between* the internal and external dipoles. The direct interaction between internal charges tends to keep the separation $\mathbf{r}=\mathbf{y}'-\mathbf{y}$ small. Using the center of mass $\mathbf{R}=(\mathbf{y}+\mathbf{y}')/2$, we can change variables to $\mathbf{y}=\mathbf{R}-\mathbf{r}/2$ and $\mathbf{y}'=\mathbf{R}+\mathbf{r}/2$, and expand the dipole–dipole interaction for small $\mathbf{r}$ as

$$\begin{aligned}
D\left(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}'\right) =& C\left(\mathbf{x}-\mathbf{R}+\frac{\mathbf{r}}{2}\right)-C\left(\mathbf{x}-\mathbf{R}-\frac{\mathbf{r}}{2}\right)-C\left(\mathbf{x}'-\mathbf{R}+\frac{\mathbf{r}}{2}\right)+C\left(\mathbf{x}'-\mathbf{R}-\frac{\mathbf{r}}{2}\right) \\
=& -\mathbf{r}\cdot\nabla_{\mathbf{R}}C(\mathbf{x}-\mathbf{R})+\mathbf{r}\cdot\nabla_{\mathbf{R}}C(\mathbf{x}'-\mathbf{R})+\mathcal{O}(r^3).
\end{aligned} \tag{VII.61}$$

To the same order

$$\begin{aligned}
e^{4\pi^2KD(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')}-1 =& -4\pi^2K\mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{x}-\mathbf{R})-C(\mathbf{x}'-\mathbf{R})\right) \\
&+8\pi^4K^2\left[\mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{x}-\mathbf{R})-C(\mathbf{x}'-\mathbf{R})\right)\right]^2+\mathcal{O}(r^3).
\end{aligned} \tag{VII.62}$$

After the change of variables $\int d^2\mathbf{y}d^2\mathbf{y}' \to \int d^2\mathbf{r}d^2\mathbf{R}$, the effective interaction becomes

$$e^{-\beta\mathcal{V}(\mathbf{x}-\mathbf{x}')} = e^{-4\pi^2 KC(\mathbf{x}-\mathbf{x}')}\Bigg\{\Bigg[1 + y_0^2 \int d^2\mathbf{r}d^2\mathbf{R}\, e^{-4\pi^2 KC(\mathbf{r})}\times$$
$$\Big( -4\pi^2 K\mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{x}-\mathbf{R}) - C(\mathbf{x}'-\mathbf{R})\right) + 8\pi^4 K^2\left[\mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{x}-\mathbf{R}) - C(\mathbf{x}'-\mathbf{R})\right)\right]^2$$
$$+ \mathcal{O}(r^3)\Big) + \mathcal{O}(y_0^4)\Bigg]\Bigg\}. \tag{VII.63}$$

Following the angular integrations in $d^2\mathbf{r}$, the term linear in $\mathbf{r}$ vanishes, while the angular average of $(\mathbf{r}\cdot\nabla_{\mathbf{R}}C)^2$ is $r^2|\nabla_{\mathbf{R}}C|^2/2$. Hence eq.(VII.63) simplifies to

$$e^{-\beta\mathcal{V}(\mathbf{x}-\mathbf{x}')} = e^{-4\pi^2 KC(\mathbf{x}-\mathbf{x}')}\times$$
$$\left[1 + y_0^2\int (2\pi r dr)e^{-4\pi^2 KC(r)}8\pi^4 K^2\,\frac{r^2}{2}\int d^2\mathbf{R}\big(\nabla_{\mathbf{R}}\left(C(\mathbf{x}-\mathbf{R}) - C(\mathbf{x}'-\mathbf{R})\right)\big)^2 + \mathcal{O}(r^4)\right]. \tag{VII.64}$$

The remaining integral can be evaluated by parts,

$$\begin{aligned}
&\int d^2\mathbf{R}\,\left[\nabla_{\mathbf{R}}\left(C(\mathbf{x}-\mathbf{R}) - C(\mathbf{x}'-\mathbf{R})\right)\right]^2 \\
=& -\int d^2\mathbf{R}\,\left(C(\mathbf{x}-\mathbf{R}) - C(\mathbf{x}'-\mathbf{R})\right)\left(\nabla^2 C(\mathbf{x}-\mathbf{R}) - \nabla^2 C(\mathbf{x}'-\mathbf{R})\right) \\
=& -\int d^2\mathbf{R}\,\left(C(\mathbf{x}-\mathbf{R}) - C(\mathbf{x}'-\mathbf{R})\right)\left(\delta^2(\mathbf{x}-\mathbf{R}) - \delta^2(\mathbf{x}'-\mathbf{R})\right) \\
&= 2C(\mathbf{x}-\mathbf{x}') - 2C(0).
\end{aligned} \tag{VII.65}$$

The short distance divergence can again be absorbed into a proper cutoff with $C(x) \to \ln(x/a)/2\pi$, and

$$e^{-\beta\mathcal{V}(\mathbf{x}-\mathbf{x}')} = e^{-4\pi^2 KC(\mathbf{x}-\mathbf{x}')}\left[1 + 16\pi^5 K^2 y_0^2 C(\mathbf{x}-\mathbf{x}')\int dr r^3 e^{-2\pi K\ln(r/a)} + \mathcal{O}(y_0^4)\right]. \tag{VII.66}$$

The second order term can be exponentiated to give an effective interaction $\beta\mathcal{V}(\mathbf{x}-\mathbf{x}') \equiv 4\pi^2 K_{\text{eff}} C(\mathbf{x}-\mathbf{x}')$, with

$$K_{\text{eff}} = K - 4\pi^3 K^2 y_0^2 a^{2\pi K}\int_a^\infty dr r^{3-2\pi K} + \mathcal{O}(y_0^4). \tag{VII.67}$$

We have thus evaluated the dielectric constant of the medium, $\varepsilon = K/K_{\text{eff}}$, perturbatively to order of $y_0^2$. However, the perturbative correction is small only as long as the integral in $r$ converges at large $r$. The breakdown of the perturbation theory for

$K < K_c = 2/\pi$, occurs precisely at the point where the free energy of an isolated vortex changes sign. This breakdown of perturbation theory is reminiscent of that encountered in the Landau–Ginzburg model for $d < 4$. Using the experience gained from that problem, we shall reorganize the perturbation series into a renormalization group for the parameters $K$ and $y_0$.