

VII. Continuous Spins at Low Temperatures

VII.A The non-linear σ -model

Previously we considered low temperature expansions for *discrete* spins (Ising, Potts, etc.), in which the low energy excitations are droplets of incorrect spin in a uniform background selected by broken symmetry. These excitations occur at small scales, and are easily described by graphs on the lattice. By contrast, for *continuous* spins, the lowest energy excitations are long-wavelength Goldstone modes, as discussed in section II.C. The thermal excitation of these modes destroys the long-range order in dimensions $d \leq 2$. For d close to 2, the critical temperature must be small, making low temperature expansions a viable tool for the study of critical phenomena. As we shall demonstrate next, such an approach requires keeping track of the interactions between Goldstone modes.

Consider unit n -component spins on the sites of a lattice, i.e.

$$\vec{s}(\mathbf{i}) = (s_1, s_2, \dots, s_n), \quad \text{with} \quad |\vec{s}(\mathbf{i})|^2 = s_1^2 + \dots + s_n^2 = 1. \quad (\text{VII.1})$$

The usual nearest neighbor Hamiltonian can be written as

$$-\beta\mathcal{H} = K \sum_{\langle \mathbf{ij} \rangle} \vec{s}(\mathbf{i}) \cdot \vec{s}(\mathbf{j}) = K \sum_{\langle \mathbf{ij} \rangle} \left(1 - \frac{(\vec{s}(\mathbf{i}) - \vec{s}(\mathbf{j}))^2}{2} \right). \quad (\text{VII.2})$$

At low temperatures, the fluctuations between neighboring spins are small and the difference in eq.(VII.2) can be replaced by a gradient. Assuming a unit lattice spacing,

$$-\beta\mathcal{H} = -\beta E_0 - \frac{K}{2} \int d^d \mathbf{x} (\nabla \vec{s}(\mathbf{x}))^2, \quad (\text{VII.3})$$

where the discrete index \mathbf{i} has been replaced by a continuous vector $\mathbf{x} \in \mathbb{R}^d$. A cutoff of $\Lambda \approx \pi$ is thus implicit in eq.(VII.3). Ignoring the ground state energy, the partition function is

$$Z = \int \mathcal{D} [\vec{s}(\mathbf{x}) \delta(s(\mathbf{x})^2 - 1)] e^{-\frac{K}{2} \int d^d \mathbf{x} (\nabla \vec{s})^2}. \quad (\text{VII.4})$$

A possible ground state configuration is $\vec{s}(\mathbf{x}) = (0, \dots, 1)$. There are $n - 1$ Goldstone modes describing the transverse fluctuations. To examine the effects of these fluctuations close to zero temperature, set

$$\vec{s}(\mathbf{x}) = (\pi_1(\mathbf{x}), \dots, \pi_{n-1}(\mathbf{x}), \sigma(\mathbf{x})) \equiv (\vec{\pi}(\mathbf{x}), \sigma(\mathbf{x})), \quad (\text{VII.5})$$

where $\vec{\pi}(\mathbf{x})$ is an $n - 1$ component vector. The unit length of the spin fixes $\sigma(\mathbf{x})$ in terms of $\vec{\pi}(\mathbf{x})$. For each degree of freedom

$$\begin{aligned} \int d\vec{s} \delta(s^2 - 1) &= \int_{-\infty}^{\infty} d\vec{\pi} d\sigma \delta(\pi^2 + \sigma^2 - 1) \\ &= \int_{-\infty}^{\infty} d\vec{\pi} d\sigma \delta\left[\left(\sigma - \sqrt{1 - \pi^2}\right)\left(\sigma + \sqrt{1 - \pi^2}\right)\right] = \int_{-\infty}^{\infty} \frac{d\vec{\pi}}{2\sqrt{1 - \pi^2}}, \end{aligned} \quad (\text{VII.6})$$

where we have used the identity $\delta(ax) = \delta(x)/|a|$. Using this result, the partition function in eq.(VII.4) can be written as

$$\begin{aligned} Z &\propto \int \frac{\mathcal{D}\vec{\pi}(\mathbf{x})}{\sqrt{1 - \pi(\mathbf{x})^2}} e^{-\frac{K}{2} \int d^d\mathbf{x} [(\nabla\vec{\pi})^2 + (\nabla\sqrt{1 - \pi^2})^2]} \\ &= \int \mathcal{D}\vec{\pi}(\mathbf{x}) \exp \left\{ - \int d^d\mathbf{x} \left[\frac{K}{2} (\nabla\vec{\pi})^2 + \frac{K}{2} \left(\nabla\sqrt{1 - \pi^2} \right)^2 + \frac{\rho}{2} \ln(1 - \pi^2) \right] \right\}. \end{aligned} \quad (\text{VII.7})$$

In going from the lattice to the continuum, we have introduced a density $\rho = N/V = 1/a^d$ of lattice points. For unit lattice spacing $\rho = 1$, but for the purpose of renormalization we shall keep an arbitrary ρ . Whereas the original Hamiltonian was quite simple, the one describing the Goldstone modes $\vec{\pi}(\mathbf{x})$, is rather complicated. In selecting a particular ground state, the rotational symmetry was broken. The nonlinear terms in eq.(VII.7) ensure that this symmetry is properly reflected when considering only $\vec{\pi}$.

We can expand the nonlinear terms for the effective Hamiltonian in powers of $\vec{\pi}(\mathbf{x})$, resulting in a series

$$\beta\mathcal{H}[\vec{\pi}(\mathbf{x})] = \beta\mathcal{H}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \cdots, \quad (\text{VII.8})$$

where

$$\beta\mathcal{H}_0 = \frac{K}{2} \int d^d\mathbf{x} (\nabla\vec{\pi})^2, \quad (\text{VII.9})$$

describes independent Goldstone modes, while

$$\mathcal{U}_1 = \int d^d\mathbf{x} \left[\frac{K}{2} (\vec{\pi} \cdot \nabla\vec{\pi})^2 - \frac{\rho}{2} \pi^2 \right], \quad (\text{VII.10})$$

is the first order perturbation when the terms in the series are organized according to powers of $T = 1/K$. Since we expect fluctuations $\langle \pi^2 \rangle \propto T$, $\beta\mathcal{H}_0$ is order of one, the two

terms in \mathcal{U}_1 are order of T ; remaining terms are order of T^2 and higher. In the language of Fourier modes,

$$\begin{aligned}\beta\mathcal{H}_0 &= \frac{K}{2} \int \frac{d^d \mathbf{q}}{(2\pi)^d} q^2 |\vec{\pi}(\mathbf{q})|^2, \\ \mathcal{U}_1 &= -\frac{K}{2} \int \frac{d^d \mathbf{q}_1 d^d \mathbf{q}_2 d^d \mathbf{q}_3}{(2\pi)^{3d}} \pi_\alpha(\mathbf{q}_1) \pi_\alpha(\mathbf{q}_2) \pi_\beta(\mathbf{q}_3) \pi_\beta(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) (\mathbf{q}_1 \cdot \mathbf{q}_3) \\ &\quad - \frac{\rho}{2} \int \frac{d^d \mathbf{q}}{(2\pi)^d} |\vec{\pi}(\mathbf{q})|^2.\end{aligned}\tag{VII.11}$$

For the non-interacting (quadratic) theory, the correlation functions of the Goldstone modes are

$$\langle \pi_\alpha(\mathbf{q}) \pi_\beta(\mathbf{q}') \rangle_0 = \frac{\delta_{\alpha,\beta} (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}')}{K q^2}.\tag{VII.12}$$

The resulting fluctuations in real space behave as

$$\langle \pi(\mathbf{x})^2 \rangle_0 = \int \frac{d^d \mathbf{q}}{(2\pi)^d} \langle |\vec{\pi}(\mathbf{q})|^2 \rangle_0 = \frac{(n-1)}{K} \int_{1/L}^{1/a} \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{q^2} = \frac{(n-1)}{K} \frac{K_d (a^{2-d} - L^{2-d})}{(d-2)}.\tag{VII.13}$$

For $d > 2$ the fluctuations are indeed proportional to T . However, for $d \leq 2$ they diverge as $L \rightarrow \infty$. This is a consequence of the Mermin–Wagner theorem on the absence of long range order in $d \leq 2$. Polyakov (1975) argued that this implies a critical temperature $T_c \sim \mathcal{O}(d-2)$ for such systems, and that an RG expansion in powers of T may provide a systematic way to explore critical behavior close to two dimensions.

To construct a perturbative RG, consider a spherical Brillouin zone of radius Λ , and divide the modes as $\vec{\pi}(\mathbf{q}) = \vec{\pi}^<(\mathbf{q}) + \vec{\pi}^>(\mathbf{q})$. The modes $\vec{\pi}^<$ involve momenta $0 < |\mathbf{q}| < \Lambda/b$, while we shall integrate over the short wavelength fluctuations $\vec{\pi}^>$ with momenta in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$. To order of T , the coarse-grained Hamiltonian is given by

$$\beta\tilde{\mathcal{H}}[\vec{\pi}^<] = V \delta f_b^0 + \beta\mathcal{H}_0[\vec{\pi}^<] + \langle \mathcal{U}_1[\vec{\pi}^< + \vec{\pi}^>] \rangle_0^> + \mathcal{O}(T^2),\tag{VII.14}$$

where $\langle \rangle_0^>$ indicates averaging over $\vec{\pi}^>$. The term proportional to ρ in eq.(VII.11) results in two contributions, one is a constant addition to free energy (from $\langle (\pi^>)^2 \rangle$), and the other is simply $\rho(\pi^<)^2$. (The cross terms proportional to $\vec{\pi}^< \cdot \vec{\pi}^>$ vanish by symmetry.) The quartic part of \mathcal{U}_1 generates 16 terms. Nontrivial contributions arise from products of two $\vec{\pi}^<$ and two $\vec{\pi}^>$. There are three types of such contributions; the first has the form

$$\begin{aligned}\langle \mathcal{U}_1^a \rangle_0^> &= 2 \times \frac{-K}{2} \int \frac{d^d \mathbf{q}_1 d^d \mathbf{q}_2 d^d \mathbf{q}_3}{(2\pi)^{3d}} (\mathbf{q}_1 \cdot \mathbf{q}_3) \\ &\quad \langle \pi_\alpha^>(\mathbf{q}_1) \pi_\alpha^>(\mathbf{q}_2) \rangle_0^> \pi_\beta^<(\mathbf{q}_3) \pi_\beta^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3).\end{aligned}\tag{VII.15}$$

The integral over the shell momentum \mathbf{q}_1 is odd and this contribution is zero. (Two similar vanishing terms arise from contractions with different indices α and β .) The next term is a renormalization of ρ , arising from

$$\begin{aligned}
\langle \mathcal{U}_1^b \rangle_0^> &= -\frac{K}{2} \int \frac{d^d \mathbf{q}_1 d^d \mathbf{q}_2 d^d \mathbf{q}_3}{(2\pi)^{3d}} (\mathbf{q}_1 \cdot \mathbf{q}_3) \\
&\quad \left\langle \pi_\alpha^>(\mathbf{q}_1) \pi_\beta^>(\mathbf{q}_3) \right\rangle_0^> \pi_\alpha^<(\mathbf{q}_2) \pi_\beta^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \\
&= \frac{K}{2} \int_0^{\Lambda/b} \frac{d^d \mathbf{q}}{(2\pi)^d} |\vec{\pi}^<(\mathbf{q})|^2 \times \int_{\Lambda/b}^{\Lambda} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{k^2}{K k^2} \\
&= \frac{\rho}{2} \int_0^{\Lambda/b} \frac{d^d \mathbf{q}}{(2\pi)^d} |\vec{\pi}^<(\mathbf{q})|^2 \times (1 - b^{-d}).
\end{aligned} \tag{VII.16}$$

(Note that in general $\rho = N/V = \int_0^{\Lambda} d^d \mathbf{q}/(2\pi)^d$.) Finally, a renormalization of K is obtained from

$$\begin{aligned}
\langle \mathcal{U}_1^c \rangle_0^> &= -\frac{K}{2} \int \frac{d^d \mathbf{q}_1 d^d \mathbf{q}_2 d^d \mathbf{q}_3}{(2\pi)^{3d}} (\mathbf{q}_1 \cdot \mathbf{q}_3) \\
&\quad \left\langle \pi_\alpha^>(\mathbf{q}_2) \pi_\beta^>(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \right\rangle_0^> \pi_\alpha^<(\mathbf{q}_1) \pi_\beta^<(\mathbf{q}_3) \\
&= \frac{K}{2} \int_0^{\Lambda/b} \frac{d^d \mathbf{q}}{(2\pi)^d} q^2 |\vec{\pi}^<(\mathbf{q})|^2 \times \frac{I_d(b)}{K},
\end{aligned} \tag{VII.17}$$

where

$$I_d(b) \equiv \int_{\Lambda/b}^{\Lambda} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{k^2} = \frac{K_d \Lambda^{d-2} (1 - b^{2-d})}{(d-2)}. \tag{VII.18}$$

The coarse-grained Hamiltonian in eq.(VII.14) now equals

$$\begin{aligned}
\beta \tilde{\mathcal{H}} [\vec{\pi}^<] &= V \delta f_b^0 + V \delta f_b^1 + \frac{K}{2} \left(1 + \frac{I_d(b)}{K} \right) \int_0^{\Lambda/b} \frac{d^d \mathbf{q}}{(2\pi)^d} q^2 |\vec{\pi}^<(\mathbf{q})|^2 \\
&\quad + \frac{K}{2} \int \frac{d^d \mathbf{q}_1 d^d \mathbf{q}_2 d^d \mathbf{q}_3}{(2\pi)^{3d}} \pi_\alpha^<(\mathbf{q}_1) \pi_\alpha^<(\mathbf{q}_2) \pi_\beta^<(\mathbf{q}_3) \pi_\beta^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) (\mathbf{q}_1 \cdot \mathbf{q}_3) \\
&\quad - \frac{\rho}{2} \int_0^{\Lambda/b} \frac{d^d \mathbf{q}}{(2\pi)^d} |\vec{\pi}^<(\mathbf{q})|^2 \times [1 - (1 - b^{-d})] + \mathcal{O}(T^2).
\end{aligned} \tag{VII.19}$$

The most important consequence of coarse graining is the change of the elastic coefficient K to

$$\tilde{K} = K \left(1 + \frac{I_d(b)}{K} \right). \tag{VII.20}$$

After rescaling, $\mathbf{x}' = \mathbf{x}/b$, and renormalizing, $\vec{\pi}'(\mathbf{x}) = \vec{\pi}^<(\mathbf{x})/\zeta$, we obtain the renormalized Hamiltonian in real space as

$$\begin{aligned} -\beta\mathcal{H}' = & -V\delta f_b^0 - V\delta f_b^1 - \frac{\tilde{K}b^{d-2}\zeta^2}{2} \int d^d\mathbf{x}' (\nabla'\pi')^2 \\ & - \frac{Kb^{d-2}\zeta^4}{2} \int d^d\mathbf{x}' (\vec{\pi}'(\mathbf{x}') \nabla' \vec{\pi}'(\mathbf{x}'))^2 + \frac{\rho\zeta^2}{2} \int d^d\mathbf{x}' \pi'(\mathbf{x}')^2 + \mathcal{O}(T^2). \end{aligned} \quad (\text{VII.21})$$

The easiest method for obtaining the rescaling factor ζ , is to take advantage of the rotational symmetry of spins. After averaging over the short wavelength modes, the spin is

$$\begin{aligned} \langle \tilde{s} \rangle_0^> &= \left\langle \left(\pi_1^< + \pi_1^>, \dots, \sqrt{1 - (\vec{\pi}^< + \vec{\pi}^>)^2} \right) \right\rangle_0^> \\ &= \left(\pi_1^<, \dots, 1 - \frac{(\vec{\pi}^<)^2}{2} - \left\langle \frac{(\vec{\pi}^>)^2}{2} \right\rangle_0^> + \dots \right) \\ &= \left(1 - \left\langle \frac{(\vec{\pi}^>)^2}{2} \right\rangle_0^> + \mathcal{O}(T^2) \right) \left(\pi_1^<, \dots, \sqrt{1 - (\vec{\pi}^<)^2} \right). \end{aligned} \quad (\text{VII.22})$$

We thus identify

$$\zeta = 1 - \left\langle \frac{(\vec{\pi}^>)^2}{2} \right\rangle_0^> + \mathcal{O}(T^2) = 1 - \frac{(n-1)}{2} \frac{I_d(b)}{K} + \mathcal{O}(T^2), \quad (\text{VII.23})$$

as the length of the coarse-grained spin. The renormalized coupling constant in eq.(VII.21) is now obtained from

$$\begin{aligned} K' &= b^{d-2}\zeta^2\tilde{K} \\ &= b^{d-2} \left[1 - \frac{n-1}{2K} I_d(b) \right]^2 K \left[1 + \frac{1}{K} I_d(b) \right] \\ &= b^{d-2} K \left[1 - \frac{n-2}{K} I_d(b) + \mathcal{O}\left(\frac{1}{K^2}\right) \right]. \end{aligned} \quad (\text{VII.24})$$

For infinitesimal rescaling, $b = (1 + \delta\ell)$, the shell integral results in

$$I_d(b) = K_d \Lambda^{d-2} \delta\ell. \quad (\text{VII.25})$$

The differential recursion relation corresponding to eq.(VII.24) is thus

$$\frac{dK}{d\ell} = (d-2)K - (n-2)K_d \Lambda^{d-2}. \quad (\text{VII.26})$$

Alternatively, the scaling of temperature $T = K^{-1}$, is

$$\frac{dT}{d\ell} = -\frac{1}{K^2} \frac{dK}{d\ell} = -(d-2)T + (n-2)K_d \Lambda^{d-2} T^2. \quad (\text{VII.27})$$

It may appear that we should also keep track of the evolution of the coefficients of the two terms in \mathcal{U}_1 under RG. In fact, spherical symmetry ensures that the coefficient of the quartic term is precisely the same as K at all orders. The apparent difference between the two is of order of $\mathcal{O}(T^2)$, and will vanish when all terms at this order are included. The coefficient of the second order term in \mathcal{U}_1 merely tracks the density of points and also has trivial renormalization.

The behavior of temperature under RG changes drastically at $d = 2$. For $d < 2$, the linear flow is away from zero, indicating that the ordered phase is unstable and there is no broken symmetry. For $d > 2$, small T flows back to zero, indicating that the ordered phase is stable. The flows for $d = 2$ are controlled by the second ordered term which changes sign at $n = 2$. For $n > 2$ the flow is towards high temperatures, indicating that Heisenberg and higher spin models are disordered. The situation for $n = 2$ is ambiguous, and it can in fact be shown that $dT/d\ell$ is zero to all orders. This special case will be discussed in more detail in the next section. For $d > 2$ and $n > 2$, there is a phase transition at the fixed point,

$$T^* = \frac{\epsilon}{(n-2)K_d \Lambda^{d-2}} = \frac{2\pi\epsilon}{(n-2)} + \mathcal{O}(\epsilon^2), \quad (\text{VII.28})$$

where $\epsilon = d - 2$ is used as a small parameter. The recursion relation at order of ϵ is

$$\frac{dT}{d\ell} = -\epsilon T + \frac{(n-2)}{2\pi} T^2. \quad (\text{VII.29})$$

Stability of the fixed point is determined by the linearized recursion relation

$$\left. \frac{d\delta T}{d\ell} \right|_{T^*} = \left[-\epsilon + \frac{(n-2)}{\pi} T^* \right] \delta T = [-\epsilon + 2\epsilon] \delta T = \epsilon \delta T, \quad \implies \quad y_t = \epsilon \quad . \quad (\text{VII.30})$$

The thermal eigenvalue, and the resulting exponents $\nu = 1/\epsilon$, and $\alpha = 2 - (2+\epsilon)/\epsilon \approx -2/\epsilon$, are independent of n at this order.

The magnetic eigenvalue can be obtained by adding a term $-\vec{h} \cdot \int d^d \mathbf{x} \vec{s}(\mathbf{x})$, to the Hamiltonian. Under the action of RG, $h' = b^d \zeta h \equiv b^{y_h} h$, with

$$b^{y_h} = b^d \left[1 - \frac{n-1}{2K} I_d(b) \right]. \quad (\text{VII.31})$$

For an infinitesimal rescaling

$$1 + y_h \delta \ell = (1 + d \delta \ell) \left(1 - \frac{n-1}{2} T^* K_d \Lambda^{d-2} \delta \ell \right), \quad (\text{VII.32})$$

leading to

$$y_h = d - \frac{n-1}{2(n-2)} \epsilon = 1 + \frac{n-3}{2(n-2)} \epsilon + \mathcal{O}(\epsilon^2), \quad (\text{VII.33})$$

which does depend on n . Using exponent identities, we find

$$\eta = 2 + d - 2y_h = \frac{\epsilon}{n-2}. \quad (\text{VII.34})$$

The exponent η is zero at the lowest order in a $4-d$ expansion, but appears at first order in the vicinity of two dimensions. The actual values of the exponents calculated at this order are not very satisfactory.