IV.E Perturbative RG (First Order)

The last section demonstrated how various expectation values associated with the Landau–Ginzburg Hamiltonian can be calculated perturbatively in powers of u. However, the perturbative series is inherently divergent close to the critical point and cannot be used to characterize critical behavior in dimensions $d \leq 4$. Wilson showed that it is possible to combine perturbative and renormalization group approaches into a systematic method for calculating critical exponents. Accordingly, we shall extend the RG calculation of Gaussian model in sec.III.G to the Landau–Ginzburg Hamiltonian, by treating $\mathcal{U} = u \int d^d\mathbf{x}\, m^4$ as a perturbation.

1. *Coarse Grain:* This is the most difficult step of the RG procedure. As before, subdivide the fluctuations into two components as,

$$\vec{m}(\mathbf{q}) = \begin{cases} \tilde{\vec{m}}(\mathbf{q}) & \text{for } 0 < q < \Lambda/b \\ \vec{\sigma}(\mathbf{q}) & \text{for } \Lambda/b < q < \Lambda \end{cases}. \tag{IV.28}$$

In the partition function,

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) \mathcal{D}\vec{\sigma}(\mathbf{q}) \exp\left\{ -\int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) \left(|\tilde{m}(\mathbf{q})|^2 + |\sigma(\mathbf{q})|^2 \right) - \mathcal{U}[\tilde{\vec{m}}(\mathbf{q}), \vec{\sigma}(\mathbf{q})] \right\}, \tag{IV.29}$$

the two sets of modes are mixed by the operator $\mathcal{U}$. Formally, the result of integrating out $\{\vec{\sigma}(\mathbf{q})\}$ can be written as

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) \exp\left\{ -\int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |\tilde{m}(\mathbf{q})|^2 \right\} \times$$
$$\exp\left\{ -\frac{nV}{2} \int_{\Lambda/b}^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \ln\left(t + Kq^2\right) \right\} \left\langle e^{-\mathcal{U}[\tilde{m},\vec{\sigma}]} \right\rangle_\sigma \equiv \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) e^{-\beta\tilde{\mathcal{H}}[\tilde{m}]}. \tag{IV.30}$$

Here we have defined the partial averages

$$\langle \mathcal{O} \rangle_\sigma \equiv \int \frac{\mathcal{D}\vec{\sigma}(\mathbf{q})}{Z_\sigma} \mathcal{O} \exp\left[-\int_{\Lambda/b}^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |\sigma(\mathbf{q})|^2 \right], \tag{IV.31}$$

with $Z_\sigma = \int \mathcal{D}\vec{\sigma}(\mathbf{q}) \exp\{-\beta\mathcal{H}_0[\vec{\sigma}]\}$, being the *Gaussian* partition function associated with the short wavelength fluctuations. From eq.(IV.30), we obtain

$$\beta\tilde{\mathcal{H}}[\tilde{m}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |\tilde{m}(\mathbf{q})|^2 - \ln\left\langle e^{-\mathcal{U}[\tilde{m},\vec{\sigma}]} \right\rangle_\sigma. \tag{IV.32}$$

The final expression can be calculated perturbatively as,

$$\ln\left\langle e^{-\mathcal{U}}\right\rangle_\sigma = -\left\langle\mathcal{U}\right\rangle_\sigma + \frac{1}{2}\left(\left\langle\mathcal{U}^2\right\rangle_\sigma - \left\langle\mathcal{U}\right\rangle_\sigma^2\right) + \cdots + \frac{(-1)^\ell}{\ell!}\times \ell^{\text{th}}\text{ cumulant of }\mathcal{U} + \cdots . \quad \text{(IV.33)}$$

The cumulants can be computed using the rules set in the previous sections. For example, at the first order we need to compute

$$\left\langle\mathcal{U}\left[\tilde{\vec{m}},\vec{\sigma}\right]\right\rangle_\sigma = u\int\frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3 d^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4)$$
$$\left\langle\left[\tilde{\vec{m}}(\mathbf{q}_1) + \vec{\sigma}(\mathbf{q}_1)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_2) + \vec{\sigma}(\mathbf{q}_2)\right]\ \left[\tilde{\vec{m}}(\mathbf{q}_3) + \vec{\sigma}(\mathbf{q}_3)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_4) + \vec{\sigma}(\mathbf{q}_4)\right]\right\rangle_\sigma \quad \text{(IV.34)}$$

The following types of terms result from expanding the product:

$$
\begin{array}{ccl}
[1] & 1 & \left\langle\tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\,\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma \\[2mm]
[2] & 4 & \left\langle\vec{\sigma}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\,\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma \\[2mm]
[3] & 2 & \left\langle\vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\,\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma \\[2mm]
[4] & 4 & \left\langle\vec{\sigma}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\,\vec{\sigma}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma \\[2mm]
[5] & 4 & \left\langle\vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\,\vec{\sigma}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma \\[2mm]
[6] & 1 & \left\langle\vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\,\vec{\sigma}(\mathbf{q}_3)\cdot\vec{\sigma}(\mathbf{q}_4)\right\rangle_\sigma
\end{array}
\quad \text{(IV.35)}
$$

The second element in each line is the number of terms with the a given 'symmetry'. The total of these coefficients is $2^4 = 16$. Since the averages $\left\langle\mathcal{O}\right\rangle_\sigma$, involve only the short wavelength fluctuations, only contractions with $\vec{\sigma}$ appear. The resulting internal momenta are integrated from Λ/b to Λ.

Term [1] has no $\vec{\sigma}$ factors and evaluates to $\mathcal{U}[\tilde{\vec{m}}]$. The second and fifth terms involve an odd number of $\vec{\sigma}$s and their average is zero. Term [3] has one contraction and evaluates to

$$-u\times 2\int\frac{d^d\mathbf{q}_1\cdots d^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1 + \cdots + \mathbf{q}_4)\frac{\delta_{jj}(2\pi)^d\delta^d(\mathbf{q}_1 + \mathbf{q}_2)}{t + Kq_1^2}\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4) =$$
$$-2nu\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}|\tilde{m}(\mathbf{q})|^2\int_{\Lambda/b}^\Lambda\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t + Kk^2}.$$
$$\text{(IV.36)}$$

Term [4] also has one contraction but there is no closed loop (the factor δ_{jj}) and hence no factor of n. The various contractions of 4 $\vec{\sigma}$ in term [6] lead to a number of terms with

no dependence on $\tilde{\vec{m}}$. We shall denote the sum of these terms by $uV\delta f_b^1$. Summing up all terms, the coarse grained Hamiltonian at order of u is given by

$$\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}] = V\left(\delta f_b^0 + u\delta f_b^1\right) + \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{\tilde{t}+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2$$
$$+ u\int_0^{\Lambda/b}\frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3}{(2\pi)^{3d}}\tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \quad , \qquad \text{(IV.37)}$$

where

$$\tilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}. \qquad\qquad \text{(IV.38)}$$

The coarse grained Hamiltonian is thus described by the same 3 parameters t, K, and u. The other two parameters in the coarse grained Hamiltonian are unchanged, i.e.

$$\tilde{K} = K, \quad \text{and} \quad \tilde{u} = u. \qquad\qquad \text{(IV.39)}$$

2. *Rescale* by setting $\mathbf{q} = b^{-1}\mathbf{q}'$, and

3. *Renormalize*, $\tilde{\vec{m}} = z\vec{m}'$, to get

$$(\beta\mathcal{H})'[m'] = V\left(\delta f_b^0 + u\delta f_b^1\right) + \int_0^{\Lambda}\frac{d^d\mathbf{q}'}{(2\pi)^d}b^{-d}z^2\left(\frac{\tilde{t}+Kb^{-2}q'^2}{2}\right)|m'(\mathbf{q}')|^2$$
$$+ uz^4b^{-3d}\int_0^{\Lambda}\frac{d^d\mathbf{q}_1' d^d\mathbf{q}_2' d^d\mathbf{q}_3'}{(2\pi)^{3d}}\vec{m}'(\mathbf{q}_1')\cdot\vec{m}'(\mathbf{q}_2')\,\vec{m}'(\mathbf{q}_3')\cdot\vec{m}'(-\mathbf{q}_1'-\mathbf{q}_2'-\mathbf{q}_3') \quad .$$
$$\text{(IV.40)}$$

The renormalized Hamiltonian is characterized by the triplet of interactions (t', K', u'), such that

$$t' = b^{-d}z^2\tilde{t}, \quad K' = b^{-d-2}z^2K, \quad u' = b^{-3d}z^4u. \qquad \text{(IV.41)}$$

As in the Gaussian model there is a fixed point at $t^* = u^* = 0$, provided that we set $z = b^{1+\frac{d}{2}}$, such that $K' = K$. The recursion relations for t and u in the vicinity of this point are given by

$$\begin{cases} t_b' = b^2\left[t + 4u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}\right] \\ u_b' = b^{4-d}u \end{cases} \quad . \qquad \text{(IV.42)}$$

While the recursion relation for u at this order is identical to that obtained by dimensional analysis; the one for t is different. It is common to convert the discrete recursion relations to continuous differential equations by setting $b = e^{\ell}$, such that for an infinitesimal $\delta\ell$,

$$t_b' \equiv t(b) = t(1+\delta\ell) = t + \delta\ell\frac{dt}{d\ell} + \mathcal{O}(\delta\ell^2) \quad , \quad u_b' \equiv u(b) = u + \delta\ell\frac{du}{d\ell} + \mathcal{O}(\delta\ell^2).$$

Expanding eqs.(IV.42) to order of $\delta\ell$, gives

$$\begin{cases} t + \delta\ell\dfrac{dt}{d\ell} = (1 + 2\delta\ell)\left(t + 4u(n+2)\dfrac{S_d}{(2\pi)^d}\dfrac{1}{t + K\Lambda^2}\Lambda^d\delta\ell\right) \\[4mm] u + \delta\ell\dfrac{du}{d\ell} = (1 + (4-d)\delta\ell)\,u \end{cases} \qquad . \qquad \text{(IV.43)}$$

The differential equations governing the evolution of t and u under rescaling are then

$$\begin{cases} \dfrac{dt}{d\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t + K\Lambda^2} \\[4mm] \dfrac{du}{d\ell} = (4-d)u \end{cases} \qquad . \qquad \text{(IV.44)}$$

The recursion relation for u is easily integrated to give $u(\ell) = u_0 e^{(4-d)\ell} = u_0 b^{(4-d)}$.

The recursion relations can be linearized in the vicinity of the fixed point $t^* = u^* = 0$, by setting $t = t^* + \delta t$ and $u = u^* + \delta u$, as

$$\frac{d}{d\ell}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & \frac{4(n+2)K_d\Lambda^{d-2}}{K} \\ 0 & 4-d \end{pmatrix}\begin{pmatrix} \delta t \\ \delta u \end{pmatrix} \qquad \text{(IV.45)}$$

In the differential form of the recursion relations, the eigenvalues of the matrix determine the relevance of operators. Since the above matrix has zero elements on one side, its eigenvalues are the diagonal elements, and as in the Gaussian model we can identify $y_t = 2$, and $y_u = 4 - d$. The results at this order are identical to those obtained from dimensional analysis on the Gaussian model. The only difference is in the eigen–directions. The exponent $y_t = 2$ is still associated with $u = 0$, while $y_u = 4 - d$ is actually associated with the direction $t = -4u(n+2)K_d\Lambda^{d-2}/K$. This agrees with the shift in the transition temperature calculated to order of u from the susceptibility.

For $d > 4$ the Gaussian fixed point has only one unstable direction associated with y_t. It thus correctly describes the phase transition. For $d < 4$ it has two relevant directions and is unstable. Unfortunately, the recursion relations have no other fixed point at this order and it appears that we have learned little from the perturbative RG. However, since we are dealing with an alternating series we can *anticipate* that the recursion relations at the next order are modified to

$$\begin{cases} \dfrac{dt}{d\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t + K\Lambda^2} - Au^2 \\[4mm] \dfrac{du}{d\ell} = (4-d)u - Bu^2 \end{cases} \qquad , \qquad \text{(IV.46)}$$

with A and B positive. There is now an additional fixed point at $u^* = (4-d)/B$ for $d < 4$. For a systematic perturbation theory we need to keep the parameter u small. Thus the new fixed point can be explored systematically only for small $\epsilon = 4 - d$; we are led to consider an expansion in the dimension of space in the vicinity of $d = 4$! For a calculation valid at $\mathcal{O}(\epsilon)$ we have to keep track of terms of second order in the recursion relation for u, but only to first order in that of t. It is thus unnecessary to calculate the term A in the above recursion relation.