Saddle point integration:

Similarly, an integral of the form

$$\mathcal{I} = \int dx \exp\big(N\phi(x)\big) \quad , \tag{4.4.11}$$

can be approximated by the maximum value of the integrand, obtained at a point $x_{\rm max}$ which maximizes the exponent $\phi(x)$. Expanding the exponent around this point gives

$$\mathcal{I} = \int dx \exp\left\{N\left[\phi(x_{\rm max}) - \frac{1}{2}|\phi''(x_{\rm max})|(x-x_{\rm max})^2 + \cdots\right]\right\} . \tag{4.4.12}$$

Note that at the maximum, the first derivative $\phi'(x_{\rm max})$, is zero, while the second derivative $\phi''(x_{\rm max})$, is negative. Terminating the series at the quadratic order results in

$$\mathcal{I} \approx e^{N\phi(x_{\rm max})} \int dx \exp\left[-\frac{N}{2}|\phi''(x_{\rm max})|(x-x_{\rm max})^2\right] \approx \sqrt{\frac{2\pi}{N|\phi''(x_{\rm max})|}} e^{N\phi(x_{\rm max})} , \tag{4.4.13}$$

where the range of integration has been extended to $[-\infty, \infty]$. The latter is justified since the integrand is negligibly small outside the neighborhood of $x_{\rm max}$.

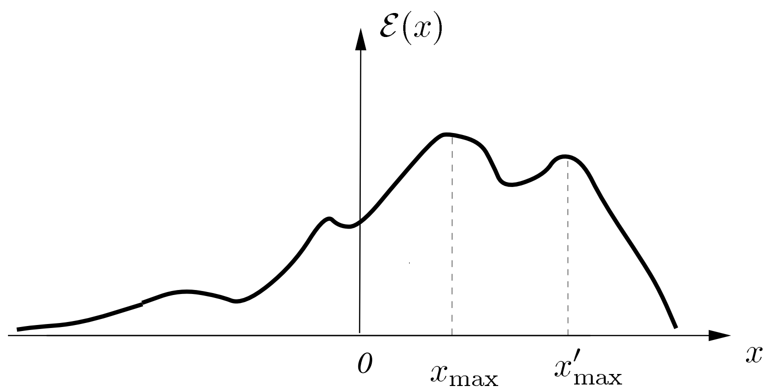

Figure 4.7: Saddle point evaluation of an 'exponential' integral.

There are two types of corrections to the above result. Firstly, there are higher order terms in the expansion of $\phi(x)$ around $x_{\rm max}$. These corrections can be looked at perturbatively, and lead to a series in powers of $1/N$. Secondly, there may be additional local maxima for the function. A maximum at $x'_{\rm max}$, leads to a similar Gaussian integral that can be added to Eq. (4.4.13). Clearly such contributions are smaller by $\mathcal{O}\big(\exp\{-N[\phi(x_{\rm max}) - \phi(x'_{\rm max})]\}\big)$. Since all these corrections vanish in the thermodynamic limit,

$$\lim_{N\to\infty} \frac{\ln \mathcal{I}}{N} = \lim_{N\to\infty}\left[\phi(x_{\rm max}) - \frac{1}{2N}\ln\left(\frac{N|\phi''(x_{\rm max})|}{2\pi}\right) + \mathcal{O}\left(\frac{1}{N^2}\right)\right] = \phi(x_{\rm max}) \quad . \tag{4.4.14}$$