1 Random-Field Ising Model

1.1 Recap: Replica formulation and dimensional-reduction

For a given realization of the random field, the Hamiltonian is

$$\beta\mathcal{H}[m;h] = \int d^d x \Big[\tfrac{K}{2}(\nabla m)^2 + \tfrac{t}{2}m^2 + u`m^4 - h(x)\,m(x)\Big], .$$

The statistical of the quenched field is assumed to satisfy $\overline{h(x)} = 0$, and $\overline{h(x)h(x')} = \sigma_h^2\,\delta^d(x-x')$.

In the replica formalism, n copies of the field are introduced, which upon averaging of Z^n, are governed by a Hamiltonian

$$\beta\mathcal{H}[\{m_\alpha\}] = \int d^d x \Big[\sum_{\alpha=1}^{n}\Big(\tfrac{K}{2}(\nabla m^\alpha)^2 + \tfrac{t}{2}(m^\alpha)^2 + u(m^\alpha)^4\Big) - \tfrac{\sigma_h^2}{2}\sum_{\alpha,\beta} m^\alpha m^\beta\Big].$$

In the $n \to 0$, the Gaussian model (with $u = 0$) leads to the expectation value

$$\overline{\tilde{m}(k)|^2} = \frac{1}{t+Kk^2} + \frac{\sigma_h^2}{(t+Kk^2)^2}\,.$$

For $k \to 0$ the disorder piece dominates, and prescribes a $1/k^4$ divergence. The perturbative integrals, with replacement of $1/k^2$ from the usual propagator with the $1/k^4$ from the disordered average, carry *two* extra powers of k^{-2} compared with the pure m^4 theory. It can then be shown that the upper critical dimension is shifted to 6, and that to all orders in perturbation theory, the exponents are the same as those of the pure theory in two lower dimensions. This elegant result, however, is in conflict with the following simple estimate of the lower critical dimension.

1.2 Imry–Ma argument

Let us consider an Ising model which in the absence of random fields has settled to one of its symmetry broken ground states (say all spins up) at very low temperature. We ask whether this ground state is stable upon the introduction of random fields. The type of instability we have in mind is flipping a large compact region of linear size L to a different state (say spins down). In the absence of random fields, such a flip will carry a cost from the unsatisfied bonds at the boundary of the flipped domain. This cost scales as σL^{d-1} in d-dimensions, where σ is an appropriate surface tension energy. Clearly thermal fluctuations are unable to overcome the energy barrier for large L and only small islands of the opposite spin will permeate the symmetry broken state at low temperatures. This is a rough justification for the possibility of symmetry breaking in pure systems in dimensions above the lower critical dimension of 1.

The presence of random fields, however, provides a different incentive for spin flips. Due to random fluctuations of random fields, a particular region of size L may have a net magnetic field that favors the opposite state. Since the domain of size L contains L^d spins, fluctuations in average may provide a net field of order $\sqrt{\sigma_h^2 L^d}$. Comparing the surface tension cost of σL^{d-1}, with a potential gain of $\sigma_h\ L^{d/2}$ indicates that large domains will then to flip in dimensions below 2, destroying long-range order. Above 2 dimensions, random fields will flip domains of typical size $\ell \sim (\sigma_h/\sigma)^{(2-2)/2}$.

Note that dimensional reduction predicts absence of long-range order in $d \leq 3$ (up 2 from lower critical dimension of 1 for pure system), yet the Imry–Ma argument would support persistence of such order in $d > 2$, including the important case of three dimensions. This conflict was eventually resolved in favor the simple Imry-Ma argument, likely due to non-perturbative effects becoming important.

Finally, note that the dimensional reduction argument also holds for vectorial spins. In the case of continuous symmetry breaking, the cost of a domain wall scales as L^{d-2} for the pure system, indicating a lower critical dimension of 2. Generalizing the Imry–Ma argument to this case, then leads to a lower critical dimension of 4 in the presence of random fields.

1.3 Rounding of discontinuous (first–order) phase transitions

An elegant heuristic extension of the Imry–Ma argument was proposed by Nihat Berker in the context of discontinuous (first–order) phase transitions. The hallmark of such systems is that upon the change of an external knob (be it a symmetry breaking magnetic field, or non-symmetry breaking parameters such as pressure or temperature), the phase transition point is marked by the coexistence of two (as in a liquid–gas system) or more (as in a disordered phases coexisting with symmetry broken states) phases. Quenched randomness is most likely to prefer one such states. This is manifestly the case for random fields in an Ising model, but random bonds in at the liquid gas transition would play a similar role.

As a concrete case, let us consider a Potts model, which for $q > 4$ in $d = 2$ is known to undergo a discontinuous transition between disordered and ordered phases. At the point of transition, the disordered phase should coexist with any one of the q symmetry broken states. However, random-bonds– while not breaking the symmetry between Potts states– are likely to locally prefer the disordered or ordered state. Invoking the Imry–Ma argument suggests that a large domain of disordered phase can invade any of the ordered states by gain of such random energy.

This generalization of the argument suggests that there can be no discontinuous phase transition in two dimensional systems with random bonds, and that first order transitions are also weakened or rounded out in higher dimensions. The heuristic argument by Berker, was proven rigorously by Aizenman and Wehr (1989); their theorem states:

> **Theorem (Aizenman & Wehr, 1989).** Let a two–dimensional spin system (with finite local state space) be perturbed by any quenched random variable that couples *linearly* to the local energy density, with translation–ergodic distribution and finite variance. Then no phase coexistence with non–vanishing order?parameter discontinuity can occur at equilibrium; i.e. all finite–temperature phase transitions are either continuous or absent.

As a consequence of this theorem, the Potts model phase transitions must become continuous for all q in $d = 2$, and that there can be no tricritical type phase diagrams.

2 Surface tension at criticality

Clearly surface tension plays a key role in stabilizing ordered phases, either due to thermal fluctuations, or energetic costs associated with quenched random impurities. It is worthwhile to discuss the behavior of surface tension close to a critical phase transition in the pure case.

Let $\sigma(T)$ denote the equilibrium interface free energy per unit area between two coexisting phases of a pure system. Close to the critical temperature T_c the singular part of the bulk free-energy density scales as $f_{\text{sing}} \sim |t|^{2-\alpha}$ with $t = (T - T_c)/T_c$. Widom's hyperscaling argument equates the excess free energy of a slab of thickness ξ (correlation length) to the cost of a single interface, yielding

$$\sigma(T)\,\xi^{\,d-1} \sim f_{\rm sing}\,\xi^{\,d} \quad\Longrightarrow\quad \sigma(T)\;\sim\;|t|^{\mu}, \qquad \boxed{\mu = (d-1)\,\nu},$$

where ν is the correlation-length exponent ($\xi \sim |t|^{-\nu}$). In two dimensions, $\nu = 1$ for the Ising universality class, so $\mu = 1$: the interface tension vanishes linearly at criticality.

It can be shown that the exact value of surface tension for the two dimensional Ising model is given by

$$\sigma(T) = 2J\left[1 - \sinh^{-2}(2\beta J)\right]^{1/2}, \qquad \beta = \frac{1}{k_{\rm B}T},$$

with $\sigma(0) = 2J$ (breaking two bonds per lattice spacing) as $T \to 0$, and $\sigma(T_c - t) = 2J\,t + \mathcal{O}(t^2)$, confirming Widom with $\mu = 1$. Surprisingly, this expression can be obtained via an uncontrolled approximation intoduced by

3 Müller–Hartmann–Zittartz estimate

Consider a square lattice with periodic boundary conditions in the x–direction. Neglecting islands and overhangs, an interface is specified by integer heights h_n ($1 \le n \le L$). For an anisotropic Ising model with couplings (K_x, K_y) each unsatisfied (+—) bond raises the energy by $2K_i$: $i = x$ for vertical bonds, $i = y$ for horizontal. Neglecting islands/overhangs, the interface has L horizontal bonds and $\sum_n |h_{n+1} - h_n|$ vertical bonds, giving

$$-\beta\mathcal{H} = -2K_y L - 2K_x \sum_{n=1}^{L} \big|h_{n+1} - h_n\big|.$$

We can construct a transfer matrix

$$\big\langle h\big|T\big|h'\big\rangle = \exp\!\big[-2K_y - 2K_x\,|h' - h|\big],$$

in the matrix form

$$T = e^{-2K_y}\begin{pmatrix} 1 & e^{-2K_x} & e^{-4K_x} & \cdots \\ e^{-2K_x} & 1 & e^{-2K_x} & \cdots \\ \vdots & & \ddots & \end{pmatrix}.$$

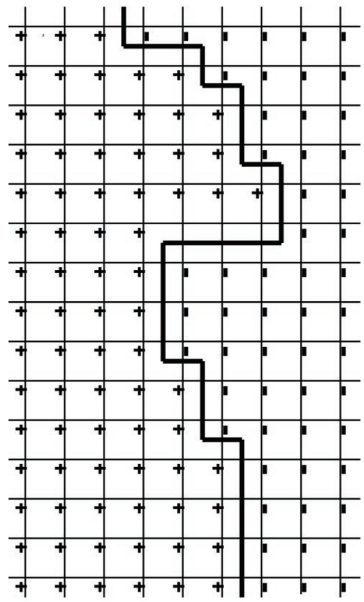

Figure 1: Square?lattice interface geometry used by Mller–Hartmann and Zittartz.

Because each row is a one–step shift of the previous row, T is diagonal in Fourier space. For wave number $k = 2\pi m/H$ $(m = 0, \ldots, H-1)$ the eigenvalue is

$$\lambda(k) = e^{-2K_y} \sum_{n=-\infty}^{\infty} e^{-2K_x|n|}\, e^{ikn}.$$

In the $H \to \infty$ limit the largest eigenvalue occurs at $k = 0$:

$$\lambda_{\max} = e^{-2K_y} \left(1 + 2 \sum_{n=1}^{\infty} e^{-2K_x n} \right) = e^{-2K_y} \coth K_x.$$

From the above result, the free energy per interface step is

$$F = -k_B T \ln \lambda_{\max} = -k_B T \Big[\ln(\coth K_x) - 2K_y \Big].$$

(A direct sum over heights produces the same result: $Z = \left[e^{-2K_y} \coth K_x\right]^L$.)

The interface tension vanishes when

$$\coth K_x = e^{2K_y},$$

which is exactly the self–duality critical line of the 2-D anisotropic Ising model. Thus long–wavelength interface fluctuations determine the critical boundary.