Scaling and Disorder in the Replica formalism

1 Recap of scaling and renormalization group

In the perspective of renormalization group (RG), critical points are controlled by a scale invariant (fixed point) Hamiltonian $-\beta\mathcal{H}^*$. Approach of a physical system to criticality is then characterized by distance from the fixed point characterized by local densities and conjugate fields. For example, onset of ferromagnetic order is described by

$$-\beta\mathcal{H} \; = -\beta\mathcal{H}^* + \int d^d x \left[h\, m(x) \; + \; t\, e(x) \; + \; u\, g(x) + \cdots\right],$$

where $m(x)$ is the magnetization density (conjugate to the magnetic field h), $e(x)$ the energy density (conjugate to the reduced temperature t), and $g(x)$ a generic (likely irrelevant) operator with coupling u.

Starting with the partition function $Z = \int \mathcal{D}m(x)\, e^{-\beta\mathcal{H}}$ and (singular part) of free energy $f(t, h, u) = -\ln Z/V$, we can obtain bulk quantities

$$M = \frac{\partial \ln Z}{\partial h} = \langle \int d^d x\; m(x)\rangle,\; E = \frac{\partial \ln Z}{\partial t} = \langle \int d^d x\; e(x)\rangle,\; G = \frac{\partial \ln Z}{\partial u} = \langle \int d^d\; x g(x)\rangle.$$

Susceptibilities follow from second derivatives, e.g. $\chi_g = (\partial G/\partial u)/V = \partial^2 f/\partial u^2$, and are related to connected two–point correlations, as in

$$\chi_g = \int d^d x \; \langle g(x)\, g(0)\rangle_c.$$

Due to scale invariance, at criticality correlations decay as a power-law, which can be parameterized as

$$\langle g(x) g(0)\rangle_c \sim |x|^{-2y_g}.$$

Away from criticality, cutting the power-law at the scale of the correlation length ξ yields

$$\chi_g \sim \int_0^\xi dr\, r^{d-1-2y_g} \sim \xi^{d-2y_g}.$$

Renormalization: Under an RG transformation that rescales distances by a factor b ($x \to b\,x$), the fields are expected to transform as

$$t \to b^{y_t} t, \qquad h \to b^{y_h}, h \qquad u \to b^{y_u} u.$$

Also, the (non-analytic) free energy density $f(h,t,u) = -\ln Z/V$ satisfies the scaling form

$$f(h,t,u) = b^{-d} f(b^{y_h} h, b^{y_t} t, b^{y_u} u).$$

Obtaining susceptibilities from second derivatives of f with respect to the fields, yields for example

$$\chi_g(h,t,u) = \frac{\partial^2 f}{\partial u^2} \sim b^{-d+2y_u} \chi_g(b^{y_h} h, b^{y_t} t, b^{y_u} u).$$

Along a particular path away from criticality, the scale parameter b is chosen to elucidate singular form of the correlation length ξ. For deviations along the thermal direction, we set $b \sim \xi$, where $\xi \sim t^{-\nu}$, with $\nu = 1/y_t$, we get

$$\chi_g \sim \xi^{-d+2y_u}.$$

Comparing this scaling to the earlier form obtained from spatial integration, $\chi_g \sim \xi^{d-2y_g}$, we obtain the identity

$$y_u + y_g = d.$$

This is general scaling relation that links the scaling dimensions of a field $u \to b^{y_u} u$ and the associated observable density $g(x) \to b^{-y_g} g(x)$, can alternatively be obtained as follows: Under the change of scale $x \to b\,x$, the critical correlations behave as

$$\langle g(x) g(0) \rangle_c \sim |x|^{-2y_g} \;\to\; |bx|^{-2y_g} \sim \langle [b^{-y_g} g(x)][b^{-y_g} g(0)] \rangle_c\,.$$

Using the above scaling of the density, $g(x) \to b^{-y_g} g(x)$, we observe

$$u \int d^d x\; g(x) \;\to\; u\; b^d\; b^{-y_g} \int d^d x\; g(x)\,, \text{ indicating } b^{y_u} = b^{d-y_g}\,.$$

2 Random bonds and the Harris Criterion

We can inquire as to what happens if one of the fields is inhomogeneously distributed in space. A relevant example is a ferromagnet with quenched impurities in which the interaction between spins varies with location. Such a *random bond system* in the field theory perspective is described by a local 'temperature' $t(x)$ that is a quenched random variable, appearing in the Hamiltonian as

$$-\beta\mathcal{H}[m(x)] = -\beta\mathcal{H}^* + \int d^d x\ [t(x)e(x) + \text{other terms}]\ .$$

Here, $e(x)$ is the local energy density and $t(x) = \bar{t} + \delta t(x)$ fluctuates in position with $\overline{\delta t(x)} = 0$ and (for uncorrelated impurity positions) $\overline{\delta t(x)\,\delta t(y)} = \sigma_t^2 \delta^d(x-y)$.

Replica Hamiltonian: To perform the disorder average, introduce n replicas:

$$\overline{Z^n} = \overline{\prod_{\alpha=1}^{n} \int \mathcal{D}m_\alpha\, e^{-\int d^d x\, [t(x)e_\alpha(x)+\cdots]}}.$$

Averaging over the Gaussian-distributed $t(x)$ with variance σ_t^2, and using the standard formula for Gaussian averages, we obtain:

$$\overline{Z^n} = \int \prod_\alpha \mathcal{D}m_\alpha \exp\left(-\int d^d x \left[\bar{t}\sum_\alpha e_\alpha(x) - \frac{\sigma_t^2}{2}\sum_{\alpha,\beta} e_\alpha(x)e_\beta(x) + \cdots\right]\right).$$

Thus, the disorder induces a *cross-replica interaction*

$$-\frac{\sigma_t^2}{2}\sum_{\alpha,\beta} e_\alpha(x)e_\beta(x).$$

(The sign of the interaction indicates preference of replicas to freeze in similar states for a given realization of random bonds.)

Scaling Analysis: We ask the question if the randomness parameter σ_t^2 is a *relevant* perturbation at the fixed point described the uniform system. At the pure fixed point, the energy density scales under RG with $x \to bx$ as:

$$e(x) \to b^{-y_e} e(x), \quad \text{with} \quad y_e = d - y_t,$$

where y_t is the RG eigenvalue of t, with $t \to b^{y_t} t$. The disorder-induced coupling σ_t^2 multiplies an operator

$$\int d^d x \, e_\alpha(x) e_\beta(x),$$

which under rescaling transforms as:

$$\int [b^d \; d^d x] \, [b^{-y_e} e_\alpha(x)][b^{-y_e} e_\beta(x)] \, ,$$

indicating that

$$\sigma_t^2 \to b^{2y_t - d} \sigma_t^2.$$

Relevance of Disorder: If $2y_t - d > 0$, the disorder grows under renormalization and is *relevant*. If $2y_t - d < 0$, the disorder shrinks and is *irrelevant*. Using the known relation between y_t and the heat capacity exponent α in the pure system:

$$\alpha = 2 - \frac{d}{y_t},$$

we can rewrite:

$$2y_t - d = y_t(2 - \frac{d}{y_t}) = y_t \alpha.$$

This is the classic **Harris criterion** which states that random bond disorder is relevant and modifies the nature of the phase transition only if the heat capacity is divergent ($\alpha > 0$).

A stronger version of the Harris Criterion states that in systems where disorder is relevant, the new critical behavior must satisfy an additional constraint: The new correlation length exponent ν' must obey $\nu' \geq \frac{2}{d}$. This ensures that fluctuations in the local critical temperature across regions of size ξ become negligible at the critical point, maintaining self-consistency of the critical scaling.

Clearly the above results can be generalized to demonstrate that uncorrelated randomness in a field $u(x)$ is only relevant if the corresponding susceptibility χ_g diverges at the fixed point.

3 Random Field Ising Model

We now consider the Random Field Ising Model (RFIM), in which a quenched random field couples linearly to the order parameter. Whereas the heat capacity exponent α may be positive or negative, the susceptibility exponent χ

is generally positive, and from previous discussion we expect random fields to modify the nature of the phase transition.

Replicated Hamiltonian with Random Fields: In the framework of the Landau-Ginzburg description, the RFIM Hamiltonian is

$$-\beta\mathcal{H}[m(x);h(x)] = \int d^dx \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 - h(x)\phi(x)\right],$$

with $h(x)$ a Gaussian random field, with

$$\overline{h(x)} = 0, \qquad \text{and} \qquad \overline{h(x)h(x')} = \sigma_h^2\,\delta^d(x-x').$$

Using the replica approach and averaging over disorder, the replicated Landau–Ginzburg Hamiltonian becomes:

$$-\beta\mathcal{H}[m_\alpha\}] = \int d^dx \left[\sum_\alpha \left(\frac{K}{2}(\nabla m_\alpha)^2 + \frac{t}{2}(m_\alpha)^2 + u(m_\alpha)^4\right) - \frac{\sigma_h^2}{2}\sum_{\alpha,\beta} m_\alpha(x)m_\beta(x)\right].$$

Disorder induces a replica–nondiagonal favorable interaction between all pairs of fields m_α, m_β.

Gaussian correlations in Fourier space: At Gaussian level (i.e., setting $u=0$), the two point correlations at wave-vector k are obtained as the inverse of

$$G^{-1}_{\alpha\beta}(k) = (Kk^2+t)\delta_{\alpha\beta} - \sigma_h^2.$$

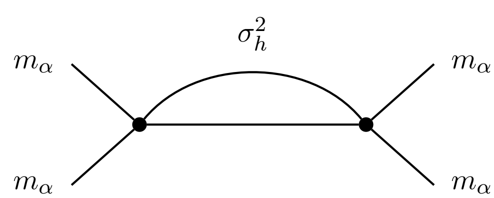
Inverting this matrix yields

$$G_{\alpha\beta}(k) = \begin{cases} \dfrac{1}{Kk^2+t} + \dfrac{\sigma_h^2}{(Kk^2+t)^2}, & \text{if } \alpha=\beta, \\ \dfrac{\sigma_h^2}{(Kk^2+t)^2}, & \text{if } \alpha\neq\beta. \end{cases}$$

- The diagonal correlator $G_{\alpha\alpha}(x-x')$ corresponds to $\overline{\langle\phi(x)\phi(x')\rangle}$, which includes both thermal and disorder-induced fluctuations.
- The off-diagonal correlator $G_{\alpha\beta}(x-x')$ for $\alpha\neq\beta$ gives $\overline{\langle\phi(x)\rangle\langle\phi(x')\rangle}$, measuring sample-to-sample fluctuations. A nonzero value indicates freezing of spins induced by the random field.

In position space, the Gaussian correlations $\langle m(x)m(0)\rangle$ can be interpreted as the sum of all phantom paths connecting points 0 and x along bonds of the lattice, carrying a factor of $\sim K$ for each bond they cross. As random walks can be regarded as fractals of dimension 2, their intersections (penalized in field theory by coupling u) are irrelevant in dimensions $d > d_{ucd} = 2 + 2 = 4$. The presence of random-fields adds to the propagator an additional $\sigma_h^2/(Kk^2+t)^2$. In position space, this corresponds to a convolution of paths from 0 to an intermediate point y, and then from y to x. Each segment terminating at y, gets a contribution from the random field $h(y)$, which upon averaging gives the factor σ_h^2. As the two random walk segments are assumed to be non-crossing, their overall structure can be regarded as that of an object of dimension $2+2=4$. To leading order in σ_h^2, the correction from intersection of paths arises when a double segment (contribution proportional to σ_h^2) intersects with a regular path of dimension 2. This leads to an upper critical dimension of $d_{ucd} = 4 + 2 = 6$ for the random field problem.

In the field theoretic approach, the quartic coupling u is marginal in $d = 4$ for the pure theory. In the RFIM, the first correction to the renormalised four-point function that contains the lowest disorder term is the diagram below:



The contribution of this diagram to the renormalization of u is proportional to

$$\int \frac{d^dp}{(2\pi)^d} \frac{\sigma_h^2}{\left(Kp^2+t\right)^2} \frac{1}{\left(Kp^2+t\right)} \xrightarrow[t\to 0]{} \sigma_h^2 \int \frac{d^dp}{p^6} \propto \Lambda^{d-6},$$

where Λ is a upper cut-off. The integral diverges logarithmically at $d = 6$, making 6 the upper-critical dimension of the RFIM.

Substituting a propagator that behaves as $\sim \sigma_h^2 k^{-4}$, in RFIM perturbation theory leads to two extra powers of k^{-2} compared with the pure m^4 theory. Consequently, the RFIM in d dimensions shares the same power counting (in all perturbative orders) as the pure model in $d-2$ dimensions: $4 \mapsto 6 = 4 + 2$. This is the field-theoretic origin of the celebrated *dimensional* *reduction by two.* Non-perturbative effects (droplets, rare regions) invalidate the reduction for $d \leq 4$, but it correctly predicts the upper critical dimension shift $4 \to 6$.