II.D Continuous symmetry breaking and Goldstone modes

For zero field, although the microscopic Hamiltonian has full rotational symmetry, the low–temperature phase does not. As a specific direction in n–space is selected for the net magnetization $\vec{M}$, there is *spontaneous symmetry breaking*, and a corresponding *long–range order* is established in which the majority of the spins in the system are oriented along $\vec{M}$. The original symmetry is still present globally, in the sense that if all local spins are rotated together (i.e. the field transforms as $\vec{m}(\mathbf{x}) \mapsto \Re\vec{m}(\mathbf{x})$), there is no change in energy. Such a rotation transforms one ordered state into an equivalent one. Since a uniform rotation costs no energy, by continuity we expect a rotation that is slowly varying in space (e.g. $\vec{m}(\mathbf{x}) \mapsto \Re(\mathbf{x})\vec{m}(\mathbf{x})$, where $\Re(\mathbf{x})$ only has long wavelength variations) to cost very little energy. Such low energy excitations are called *Goldstone modes*. These collective modes appear in any system with a *broken continuous symmetry*.[5] Phonons in a solid provide a familiar example of Goldstone modes, corresponding to the breaking of translation and rotation symmetries by a crystal structure.

The Goldstone modes appearing in diverse systems share certain common characteristics. Let us explore the origin and behavior of Goldstone modes in the context of *superfluidity*. In analogy to Bose condensation, the superfluid phase has a macroscopic occupation of a single quantum ground state. The order parameter,

$$\psi(\mathbf{x}) \equiv \psi_{\Re}(\mathbf{x}) + i\psi_{\Im}(\mathbf{x}) \equiv |\psi(\mathbf{x})|e^{i\theta(\mathbf{x})}, \tag{II.21}$$

can be roughly regarded as the ground state component (overlap) of the actual wavefunction in the vicinity of $\mathbf{x}$.[6] The phase of the wavefunction is not an observable quantity and should not appear in any physically measurable probability. This observation constrains the form of the effective coarse grained Hamiltonian, leading to an expansion

$$\beta\mathcal{H} = \beta F_0 + \int d^d\mathbf{x} \left[\frac{K}{2}|\nabla\psi|^2 + \frac{t}{2}|\psi|^2 + u|\psi|^4 + \cdots\right]. \tag{II.22}$$

Equation (II.22) is in fact equivalent to the Landau–Ginzburg Hamiltonian with $n = 2$, as can be seen by changing variables to the two component field $\vec{m}(\mathbf{x}) \equiv (\psi_{\Re}(\mathbf{x}), \psi_{\Im}(\mathbf{x}))$. The

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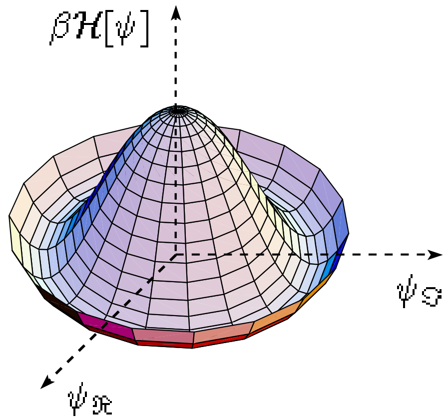
[5] There are no Goldstone modes when a discrete symmetry is broken, since it is impossible to produce slowly varying rotations from one state to an equivalent one.

[6] A more rigorous derivation proceeds by second quantization of the Hamiltonian for interacting bosons, and is beyond our current scope.



II.7. The Landau–Ginzburg Hamiltonian of a superfluid for $t < 0$.

superfluid transition is signaled by the onset of a finite value of ψ for $t < 0$. The Landau–Ginzburg Hamiltonian (for a uniform ψ) has the shape of a wine bottle (or Mexican hat) for $t < 0$.

Minimizing this function sets the magnitude of ψ, but not fix its phase θ. Now consider a state with a slowly varying phase, i.e. with $\psi(\mathbf{x}) = \overline{\psi} e^{i\theta(\mathbf{x})}$. Inserting this form into the Hamiltonian yields an energy

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \frac{\overline{K}}{2} \int d^d\mathbf{x} (\nabla\theta)^2, \tag{II.23}$$

where $\overline{K} = K\overline{\psi}^2$. As in the case of phonons we could have guessed the above form by appealing to the invariance of the energy function under a uniform rotation: Since a transformation $\theta(\mathbf{x}) \mapsto \theta(\mathbf{x}) + \theta_0$ should not change the energy, the energy density can only depend on gradients $\theta(\mathbf{x})$, and the first term in the expansion leads to Eq.(II.23). The reasoning based on symmetry does not give the value of the *stiffness parameter*. By starting from the Landau–Ginzburg form which incorporates both the normal and superfluid phases we find that $\overline{K}$ is proportional to the square of the order parameter, and vanishes (softens) at the critical point as $\overline{K} \propto \overline{\psi}^2 \propto t$.

We can decompose the variations in phase of the order parameter into independent normal modes by setting (in a region of volume V)

$$\theta(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} \theta_{\mathbf{q}}. \tag{II.24}$$

Taking advantage of translational symmetry, Eq.(II.23) then gives

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \frac{\overline{K}}{2} \sum_{\mathbf{q}} q^2 |\theta(\mathbf{q})|^2. \tag{II.25}$$

We can see that, as in the case of phonons, the energy of a Goldstone mode of wavenumber $\mathbf{q}$ is proportional to q^2, and becomes very small at long wavelengths.