Symmetry mixing in dual models as a diagnostic for Lieb-Schultz-Mattis anomalies

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Lieb-Schultz-Mattis theorems place constraints on the ground states of quantum many-body systems, allowing for an analytic proof of certain types of quantum order. However, upon implementing a transformation to a dual model, such theorems may no longer apply, and the nature of the symmetries and phases in the model can change drastically. I review recent results which show how the trivialization of these theorems manifests itself as a mixing of symmetries in the dual model and consider the specific example of the quantum spin-1/2 XYZ model.

I. INTRODUCTION

Dualities form the basis for many aspects of statistical mechanics, condensed matter, and the study of phase transitions. Indeed, many exact results describing phase transitions can be obtained from mapping between dual models; for example, the 2D Ising model's exact transition point can be determined by a Kramers-Wannier (KW) self-duality, and the 1+1D quantum Ising model can be exactly solved using the Jordan-Wigner (JW) duality [1, 2]. Moreover, these dualities can provide important insights when considering exotic phases of quantum matter, such as topological and symmetry-protected topological orders. Understanding such dualities and how they are related is therefore paramount to understanding phases of matter. In addition to such dualities, the existence of Lieb-Schultz-Mattis (LSM) theorems gives us another non-perturbative handle on describing equilibrium phases [2, 3]. More specifically, LSM theorems ensure that certain local quantum lattice Hamiltonians have ground states which either break a symmetry, support topological order, or are gapless. In particular, LSM theorems apply to certain models with some combination of crystalline and internal symmetries, allowing us to constrain their zero-temperature phase diagrams.

Under a duality mapping, LSM theorems can become irrelevant; for instance, a phase which obeys an LSM theorem may become a unique gapped ground state after a duality. The specific way in which this trivialization occurs was recently elucidated in Ref. [4]. In the present work, we closely follow the treatment in this paper and condense it into a more conceptual, less technical format.

II. GAUGING DUALITIES

We begin by framing both the KW and JW mappings as dualities induced by gauging a $\mathbb{Z}_2$ symmetry of the original model. More specifically, consider some $\mathbb{Z}_2$ symmetry which acts on spin-1/2s located on the sites of a 1D chain. This global symmetry can be broken into a product of local operations:

$$U = \prod_j U_j \ . \tag{1}$$

The Hamiltonian only respects this symmetry if it is applied to all sites at once. However, we can promote this symmetry to a *local* version at the cost of introducing new degrees of freedom. The intuition behind this is that applying U_j to a single site of the lattice will generically alter interaction terms which couple different sites, breaking the symmetry. However, if we place new degrees of freedom (e.g. spins or fermions) on the links of our lattice which also transform under U_j, they can mediate the interaction between the original spins and preserve the symmetry of the Hamiltonian. In essence, these new degrees of freedom keep track of local "rotations" and ensure they do not alter interactions.

After "gauging" in this way, we have a new theory which includes our original (matter) degrees of freedom and our new (gauge) degrees of freedom. To implement a duality, we need to eliminate (in the field theory language, integrate out) the matter spins, converting our original Hamiltonian into a dual version on the gauge degrees of freedom. At this point, we have a new Hamiltonian which serves as a dual to the original one in the following sense: given some operator O in the original theory, we can find duals $O^\vee$ which satisfy

$$\langle O\rangle_Z = \langle O^\vee\rangle_{Z^\vee} \ , \tag{2}$$

where $Z \propto e^{-\beta H}$ is some quantum partition function (with $Z^\vee$ defined similarly). Note that this may only apply to some subset of operators in the theory.

A. Kramers-Wannier duality

Let us now formalize the above intuition using the Kramers-Wannier duality as an example. As stated, we consider a chain of spin-1/2 particles located on sites j of a lattice Λ of length $2N$. We will focus on the $\mathbb{Z}_2^z$ spin flip symmetry generated by

$$U_z \equiv \prod_j \sigma_j^z \ , \tag{3}$$

where σ_j^a is the ath Pauli operator acting on site j. For the sake of simplicity, we will use periodic boundary conditions such that $\sigma_{j+2N}^a = \sigma_j^a$. We now define the following bond algebra:

$$\mathfrak{B}_b \equiv \left\langle \sigma_j^z, \sigma_j^x \sigma_{j+1}^x \,|\, j \in \Lambda \right\rangle \, . \tag{4}$$

These operators generate the set of all operators which respect the $\mathbb{Z}_2^z$ flip symmetry. As such, they can all be block diagonalized according to the symmetry eigenspaces, and we can similarly break the Hilbert space into even and odd sectors under the symmetry: $\mathcal{H}_b = \mathcal{H}_{b+} \oplus \mathcal{H}_{b-}$.

We can now proceed to gauge our model. The first step is to introduce new spins $\tau_{j^*}^a$ on the links of the lattice, where $j^* \equiv j + 1/2$ labels the link, in such a way that we obtain a local $\mathbb{Z}_2^z$ symmetry (for simplicity's sake, we again require that the spins obey periodic boundary conditions). The gauge symmetry will now be generated by $2N$ local unitaries known as the Gauss operators

$$G_j \equiv \tau_{j^*-1}^x \sigma_j^z \tau_{j^*}^x \, . \tag{5}$$

Intuitively, this is the original $U_j = \sigma_j^z$ along with X rotations on the neighboring gauge spins. These new rotations allow the link spins to keep track of local site flips and preserve the form of interactions. In particular, the interactions from the bond algebra Eq. (4) must be modified:

$$\mathfrak{B}_{bb'} \equiv \left\langle \sigma_j^z, \sigma_j^x \tau_{j^*}^z \sigma_{j+1}^x \,|\, j \in \Lambda \right\rangle \, . \tag{6}$$

The interaction terms now have a gauge spin to mediate the interaction and "undo" local basis changes, allowing for gauge symmetric interactions. Indeed, note that all elements of the extended bond algebra commute with all Gauss operators G_j, implying we have successfully gauged our global symmetry.

In field theory, we interpret these additional degrees of freedom as redundancies in our description of the theory rather than veritable physical degrees of freedom. This implies that the gauge symmetry is not a real symmetry at all but rather a mapping between states which are gauge equivalent: physically, they represent the same state. We can choose one of these equivalent states to be the representative of the physical state by requiring that $G_j |\psi_{phys}\rangle = |\psi_{phys}\rangle$, i.e. physical states are invariant under any gauge transformation. Making this identification reduces our Hilbert space back to its original size, eliminating our extra degrees of freedom.

However, recall that we want to finish the duality mapping with a theory defined only on the gauge spins and eliminate the matter spins. Thus, we must ensure that this projection onto the gauge invariant subspace eliminates our original spins from the model. This can be done by constructing a unitary $U_{bb'}$ (explicitly written in Ref. [4]) which localizes the Gauss operators on the matter sites:

$$\begin{aligned} U_{bb'} \sigma_j^x U_{bb'}^\dagger &= \sigma_j^x \quad & U_{bb'} \sigma_j^z U_{bb'}^\dagger &= \tau_{j^*-1}^x \sigma_j^z \tau_{j^*}^x \\ U_{bb'} \tau_{j^*}^x U_{bb'}^\dagger &= \tau_{j^*}^x \quad & U_{bb'} \tau_{j^*}^z U_{bb'}^\dagger &= \sigma_j^x \tau_{j^*}^z \sigma_{j+1}^x \, , \end{aligned} \tag{7}$$

and $U_{bb'} G_j U_{bb'}^\dagger = \sigma_j^z$. After this basis change, projecting to the gauge-invariant subspace corresponds to setting $\sigma_j^z = 1$ on all sites. After this procedure, we have a new dual theory defined on the same form of Hilbert space as before. The dual bond algebra is

$$\mathfrak{B}_{b'} \equiv \left\langle \tau_{j^*-1}^{x\vee} \tau_{j^*}^{x\vee}, \tau_{j^*}^{z\vee} \,|\, j^* \in \Lambda^* \right\rangle \, , \tag{8}$$

where Λ^* is the dual lattice of links. Notice that we have added the extra superscript $\vee$ to draw attention to the fact that these are dual operators. The bond algebra is invariant under a new $\mathbb{Z}_2^{z\vee}$ spin flip rotation

$$U_z^\vee = \prod_{j^*} \tau_{j^*}^{z\vee} \, . \tag{9}$$

Notice that the duality mapping has sent bond operators to site operators (and vice versa) while maintaining a spin flip symmetry. This is exactly the Kramers-Wannier duality as used in the Ising model to map between ordered and disordered phases. In particular, it fixes the critical point of the transverse field Ising model to be at the fixed point of the mapping in parameter space. In passing, we note that symmetry actually constrains the subspace of the Hilbert space under which the duality holds; in particular, we must map between states that are even under the spin flip, restricting us to $\mathcal{H}_{b+}$ and $\mathcal{H}_{b'+}^\vee$.

B. Jordan-Wigner duality

Using the tools we learned from the previous subsection, we can now answer the question: what if our gauge degrees of freedom aren't spins? Intuitively, they only need to be able to carry the information of local spin flips and mediate interactions in a consistent way. If we instead decide to use fermions as our gauge particles, we will derive the Jordan-Wigner transformation under the gauging procedure outlined above. In particular, we define Majorana fermions on the dual lattice Λ^* which obey the Clifford algebra

$$\{\alpha_{i^*}, \alpha_{j^*}\} = \{\beta_{i^*}, \beta_{j^*}\} = 2\delta_{i^*j^*} \quad \{\alpha_{i^*}, \beta_{j^*}\} = 0 \, . \tag{10}$$

We also define a $\mathbb{Z}_2^F$ fermion parity symmetry generated by

$$P_F \equiv \prod_{j^*} i\beta_{j^*}\alpha_{j^*} \, . \tag{11}$$

With fermions, boundary conditions become a little tricky, so we will leave the choice of periodic or antiperiodic arbitrary for now:

$$\alpha_{j^*+2N} = (-1)^f \alpha_{j^*} \, , \tag{12}$$

and similarly for β_{j^*}, where $f = 0, 1$ depending on our choice. In direct analogy to the bosonic case, we will use these fermions to gauge the original spin flip symmetry such that the Gauss operators are

$$G_j = i\beta_{j^*-1} \sigma_j^z \alpha_{j^*} \, . \tag{13}$$

As before, these generate local versions of our original spin flip with the caveat that $\prod_j G_j = (-1)^f P_F U_z$ rather than just U_z in the bosonic case. As before, we extend the bond algebra to be invariant under any of these Gauss operators:

$$\mathfrak{B}_{bf} \equiv \left\langle \sigma_j^z, \sigma_j^x (i\beta_{j^*}\alpha_{j^*})\sigma_{j+1}^x \,|\, j \in \Lambda \right\rangle \ . \tag{14}$$

As before, the interactions are now mediated by gauge fermions. Following our recipe, we now need to localize the Gauss operators on matter spins and then project into the gauge invariant subspace. Ref. [4] constructs the explicit localizing unitary U_{bf} which implements

$$\begin{aligned} U_{bf}\sigma_j^x U_{bf}^\dagger = \sigma_j^x \quad U_{bf}\sigma_j^z U_{bf}^\dagger = i\beta_{j^*-1}\sigma_j^z \alpha_{j^*} \\ U_{bf}\beta_{j^*} U_{bf}^\dagger = \beta_{j^*}\sigma_{j+1}^x \quad U_{bf}\alpha_{j^*} U_{bf}^\dagger = \sigma_j^x \alpha_{j^*} \ , \end{aligned} \tag{15}$$

and $U_{bf} G_j U_{bf}^\dagger = \sigma_j^z$. Our final bond algebra is

$$\mathfrak{B}_f \equiv \left\langle i\beta_j^\vee \alpha_j^\vee, i\beta_{j+1}^\vee \alpha_j^\vee \,|\, j \in \Lambda \right\rangle \ . \tag{16}$$

Note two changes: first, as before, we have written $\vee$ to draw attention to the duality of the Hilbert space. Second, *we are no longer working on the dual lattice.* Rather, we have decided to associate $\beta_{j^*} \to \beta_{j+1}^\vee$ and $\alpha_{j^*} \to \alpha_j^\vee$. This is a convention which amounts to doing a "half translation" on the Hilbert space for a single species of Majorana. The dual bond algebra generates all operators symmetric under fermion parity:

$$P_F^\vee \equiv \prod_j i\beta_j^\vee \alpha_j^\vee \ . \tag{17}$$

The duality shown here exactly implements the well-known Jordan-Wigner mapping used to e.g. exactly solve the transverse field Ising model and the quantum XY model. In passing, we note that (just as in the KW case), the choice of boundary conditions f restricts which subspaces of the Hilbert space (even or odd under fermion parity) are related via duality.

C. Triality

Although we won't go into full detail here, the 3 different bond algebras discussed above (b, b', f) can all be related to each other via gauging the $\mathbb{Z}_2$ symmetry present in the original model. In particular, the remaining maps proceed as follows:

- The map $\mathfrak{B}_{b'} \to \mathfrak{B}_b$ is implemented by gauging the $\mathbb{Z}_2^{z\vee}$ symmetry with spins on the links.
- The map $\mathfrak{B}_{b'} \to \mathfrak{B}_f$ is implemented by gauging the $\mathbb{Z}_2^{z\vee}$ symmetry with Majoranas on the links.
- The map $\mathfrak{B}_f \to \mathfrak{B}_{b'}$ is implemented by gauging the $\mathbb{Z}_2^F$ symmetry with spins on the links.
- The map $\mathfrak{B}_f \to \mathfrak{B}_b$ is implemented by gauging the $\mathbb{Z}_2^F$ symmetry with spins on the sites.

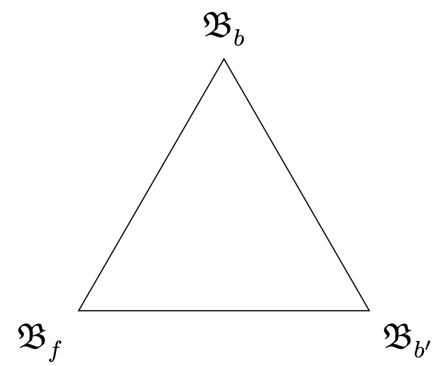

FIG. 1: Triality induced by gauging. Any bond algebra can be reached from any other by an appropriate gauging of the $\mathbb{Z}_2$ symmetry present in the model. (Image originally from [4])

The last two bullets are related by a simple translation of βs by a lattice site. This shift exchanges links for sites, so intuitively it follows that the resulting bosonic models are related by a KW transformation (which exchanges links and sites). A diagrammatic understanding of this triality is shown in Fig. 1.

Finally, we note that having a consistent triality between symmetry sectors of our Hamiltonian requires that $f = 1$ for our periodic boundary conditions in the two bosonic models.

III. LSM ANOMALIES UNDER GAUGING

The previous section elucidated the fate of bond algebras and $\mathbb{Z}_2$ symmetries under dualities induced by gauging. We can now address the question of how additional spatial symmetries (translation and reflection) manifest themselves in dual models.

Let's impose these spatial symmetries using the following definitions: a translation maps sites using $t(j) \equiv j+1 \mod 2N$ and site-centered reflection maps sites using $r(j) \equiv 2N - j \mod 2N$. Then the total spatial group is given by the semi-direct product of reflection acting on translations:

$$G_{spa} \equiv \mathbb{Z}_{2N}^t \rtimes \mathbb{Z}_2^r \ , \tag{18}$$

with the semidirect action $rt = t^{2N-1}r$. The internal symmetries we impose are given by a global X flip and a global Y flip, the product of which generates our global Z flip up to a sign:

$$G_{int} \equiv \mathbb{Z}_2^x \times \mathbb{Z}_2^y \supset \mathbb{Z}_2^z \ . \tag{19}$$

Finally, our periodic boundary conditions ensure that the total symmetry group is just the direct product of these two: $G_{tot} \equiv G_{spa} \times G_{int}$.

We can now see the applicability of an LSM theorem to our system. Although the global symmetry actions commute because of our even number of sites and therefore form a linear representation of $\mathbb{Z}_2^x \times \mathbb{Z}_2^y$, the Pauli algebra ensures that our symmetries are realized *projectively* on each site. That is, the representation of our group is faithful up to overall phases (-1 in this case).

In particular, we consider the following theorems from [4]:

Theorem 1 *Given a 1D chain with* $G_{tot} \equiv \mathbb{Z}^t_{|\Lambda|} \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y$*, if the unit cell of translation hosts a half-integer spin representation (i.e. projective), the ground state cannot be simultaneously gapped, non-degenerate, and* G_{tot} *symmetric.*

Theorem 2 *Given a 1D chain with* $G_{tot} \equiv \mathbb{Z}_2^r \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y$*, if the center of reflection hosts a half-integer spin representation (i.e. projective), the ground state cannot be simultaneously gapped, non-degenerate, and* G_{tot} *symmetric.*

These translation and reflection LSM theorems can be reinterpreted as mixed 't Hooft anomalies between spatial and internal symmetries. Put more plainly, one cannot gauge G_{int} while still preserving G_{spa}, which puts constraints on the ground state spectrum. However, one can note that subgroups $H_{int} \subset G_{int}$ can be "non-anomalous" in the sense that they can be consistently gauged without breaking the remaining symmetries.

This is the main result of [4], which shows that $\mathbb{Z}_2^z \subset \mathbb{Z}_2^x \times \mathbb{Z}_2^y$ can be gauged non-anomalously and leads to detectable results in the corresponding dual models. More specifically, the remaining symmetries $\mathbb{Z}_2^x \times \mathbb{Z}_2^y / \mathbb{Z}_2^z$ and G_{spa} mix into a semidirect product after implementing the duality. This mixing can then be used as a diagnostic for when LSM theorems apply in dual models.

A. LSM after KW

To be specific, let's consider the KW mapping. When extending our Hilbert space, the guiding principle in extending symmetries is that the Gauss operators are covariant under spatial symmetries. We identify $t(j^*) \equiv j^* + 1 \mod 2N$ and $r(j^*) \equiv 2N - j^* \mod 2N$ as one would expect, so that these transformations map $G_j \to G_{t(j)}$ and $G_j \to G_{r(j)}$. The dual images of the spatial symmetries are

$$G^\vee_{spa} \equiv \mathbb{Z}^t_{2N} \rtimes \mathbb{Z}_2^r \, . \tag{20}$$

The crucial difference here is that reflection is now *link*-centered rather than site-centered.

The more interesting case is how the internal symmetries dualize. We have to break $U_x = \prod_j \sigma_j^x$ and $U_y = \prod_j \sigma_j^y$ into a product of bond algebra generators to identify how they map, but it's clear that there is an ambiguity in doing so. Consider the X case, where we can write the unitary equivalently as

$$U_x = \prod_{j\in o} \sigma_j^x \sigma_{j+1}^x = \prod_{j \in e} \sigma_j^x \sigma_{j+1}^x \, , \tag{21}$$

where o and e correspond to the sets of odd and even sites respectively. As such, we get *two* different symmetry operators after dualizing:

$$U_o^\vee \equiv \prod_{j^* \in o} \tau_{j^*}^{z\vee}, \quad U_e^\vee \equiv \prod_{j^* \in e} \tau_{j^*}^{z\vee} \, . \tag{22}$$

Note that the product of these two symmetries generates the dual Z spin flip $U_z^\vee$. One can also show that U_y generates the same set of operators. Thus, the internal symmetries in our KW dual model are $G^\vee_{int} = \mathbb{Z}_2^o \times \mathbb{Z}_2^e$. However, these symmetries have a property which wasn't present before: *they transform nontrivially under spatial symmetries.* In particular, a translation by one site or a link-centered reflection will permute odd and even sites. This implies that the full dual symmetry group is

$$G^\vee_{tot} = G^\vee_{spa} \ltimes G^\vee_{int} \, . \tag{23}$$

This is the symmetry mixing which we can use as a diagnostic of the LSM anomaly in the original model.

Note that the internal symmetries are now realized via a linear representation (there are no Paulis present besides Z), so the LSM theorems above do not apply in the dual models. More precisely, the unit cell of translation has two spin-1/2 sites, and there are no invariant sites under reflection. Intuitively, the symmetry mixing can be interpreted as the "cost" for trivializing the LSM anomaly.

In passing, we note that another interpretation touched on in [4] describes how spatially modulated symmetries, like that associated with dipole moment conservation, appear after gauging. In other words, an LSM anomaly is the cost of changing a spatially modulated symmetry to a spatially uniform symmetry.

B. LSM after JW

The fate of the LSM anomaly under a JW transformation follows the same story as the KW case except for a couple of details. First, notice that the final Majoranas are defined on the direct lattice instead of the dual lattice, so the reflection is still site-centered instead of link-centered. Second, reflection and translation mix nontrivially with fermion parity. One can show that reflecting twice or translating by $2N$ gives

$$(U_r^\vee)^2 = -P_F \quad (U_t^\vee)^{2N} = (P_F)^f \, . \tag{24}$$

We will leave the $\vee$ symbols implicit on fermionic objects to avoid clutter. The fermion parity group therefore acts as a central extension of the spatial symmetry group, a (partial) result of our boundary conditions $f = 1$ forced on us by choosing periodic boundaries in the spin chains.

After dualizing our internal symmetries, we once again get operators acting on even or odd pairs of sites:

$$U_o^\vee \equiv \prod_{j\in o} i\alpha_j\beta_{j+1}, \quad U_e^\vee \equiv \prod_{j\in e} i\beta_{j-1}\alpha_j \, , \tag{25}$$

where again the product of the two generates the relevant $\mathbb{Z}_2$ symmetry P_F. The internal symmetry group $G_{int}^{\vee F} = \mathbb{Z}_2^o \times \mathbb{Z}_2^e$ mixes nontrivially with spatial symmetries as before. Moreover, we have extra structure in the group because of the mixing with fermion parity.

Once again, we need to build the full symmetry out of the semidirect product of spatial and internal symmetries. However, because both include fermion parity as a subgroup, the true total group must quotient one out:

$$G_{tot}^{\vee F} = \left(G_{spa}^{\vee F} \ltimes G_{int}^{\vee F}\right)/\mathbb{Z}_2^F \, . \tag{26}$$

Note the extra structure beyond a simple semidirect product in this dual model. Once again, the local representations of the internal symmetries are not projective, so neither of the LSM theorems apply. This can be interpreted just as before: we exchange an LSM anomaly for symmetry mixing (or dipole conservation).

IV. SPIN-1/2 XYZ CHAIN

As a concrete example of the concepts discussed above, Ref. [4] analyzes the quantum spin-1/2 XYZ chain and its ground state phase diagram using the KW and JW dualities. While the full Hamiltonian can be dualized, we will restrict ourselves to have no Z components, so our Hamiltonian is

$$\begin{aligned} H_b =& J_1 \sum_j (\Delta_x \sigma_j^x \sigma_{j+1}^x + \Delta_y \sigma_j^y \sigma_{j+1}^y) \\ &+ J_2 \sum_j (\Delta_x \sigma_j^x \sigma_{j+2}^x + \Delta_y \sigma_j^y \sigma_{j+2}^y) \, , \end{aligned} \tag{27}$$

with the total lattice size $2N$ a multiple of 4 for simplicity. We will restrict all parameters to be positive (such that the couplings are antiferromagnetic) and will define $J \equiv J_2/J_1$ and $\Delta \equiv \Delta_y/\Delta_x$. Finally, for simplicity, we will restrict $J \leq 1/2$. The phase diagram of this model is shown in Fig. 2a. In the bottom left and right corners, the fixed points are the Ising antiferromagnet polarized along x and y respectively. Along the Majumdar-Ghosh line, there are 2 degenerate ground states which are exactly solvable: they are constructed from singlet states between neighboring pairs of spins. The degeneracy arises from choosing odd or even pairs to form singlets.

Each of these fixed points defines a phase (Neel_x, Neel_y, and dimer) which is robust to local perturbations in the thermodynamic limit. However, note that all of these phases break the full symmetry group G_{tot} down to some subgroup, whether it be internal or spatial. Thus, none of the phase transitions follow Landau's paradigm of symmetry breaking in the usual sense. Rather, each transition is a deconfined quantum critical point (DQCP) where defects of one phase nucleate and generate the order of the new phase. This result follows from the LSM theorems mentioned earlier which imply that all gapped phases must have degenerate ground states. All phases break translation symmetry, so we distinguish them by how they break the remaining symmetries: $\mathbb{Z}_2^r \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y$. The Neel phases break their respective internal symmetry, while the dimer phase breaks the reflection symmetry, and order parameters can be defined for each (see [4]) to distinguish the breaking patterns.

A. KW dual of XYZ model

After performing the KW duality, our new Hamiltonian is

$$\begin{aligned} H_{b'} =& J_1 \sum_{j^*} (\Delta_x \tau_{j^*}^{z\vee} - \Delta_y \tau_{j^*-1}^{x\vee} \tau_{j^*}^{z\vee} \tau_{j^*+1}^{x\vee}) \\ &+ J_2 \sum_{j^*} (\Delta_x \tau_{j^*}^{z\vee} \tau_{j^*+1}^{z\vee} \\ &+ \Delta_y (\tau_{j^*-1}^{x\vee} \tau_{j^*}^{z\vee} \tau_{j^*+1}^{x\vee})(\tau_{j^*}^{x\vee} \tau_{j^*+1}^{z\vee} \tau_{j^*+2}^{x\vee})) \, . \end{aligned} \tag{28}$$

The original $\mathbb{Z}_2^r \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y$ symmetry, as we have seen, will mix under KW. In this case, the final symmetry group is $D_8 \simeq \mathbb{Z}_2^r \ltimes (\mathbb{Z}_2^o \times \mathbb{Z}_2^e)$.

The location of phase transitions in parameter space is invariant under duality, but the nature of the phases is not. Thus, let's analyze the new phase diagram, as shown in Fig. 2b. The Neel_x phase is now replaced by the paramagnet defined by the fixed point wavefunction $|\downarrow\rangle^{\otimes 2N}$. We now have a unique gapped state which is symmetric under all elements of D_8. This would not be possible unless our duality had trivialized the LSM anomaly in exchange for mixing the remaining symmetries.

The Neel_y phase is replaced by the cluster symmetry-protected topological (SPT) phase whose fixed point wavefunction is defined implicitly by $\tau_{j^*-1}^{x\vee} \tau_{j^*}^{z\vee} \tau_{j^*+1}^{x\vee} |\text{cluster}\rangle = |\text{cluster}\rangle$. Such a phase is referred to as topological because its degeneracy depends on the topology of our space, i.e. whether our boundary conditions are open or closed. On our periodic chain, the ground state is unique and symmetric under D_8. Once again, this is only possible because of the trivialization of the LSM anomalies under gauging. One may ask why there is a transition between the paramagnet and the SPT if no symmetry is broken. This transition lies outside the Landau paradigm as well; as long as our Hamiltonian always respects the symmetry $\mathbb{Z}_2^o \times \mathbb{Z}_2^e$ which protects the SPT, there must be a phase transition to leave the SPT. This is the sense in which it is "protected" by symmetry.

Finally, the dimer phase is mapped to a phase which spontaneously breaks the D_8 group down to $\mathbb{Z}_2$ with 4

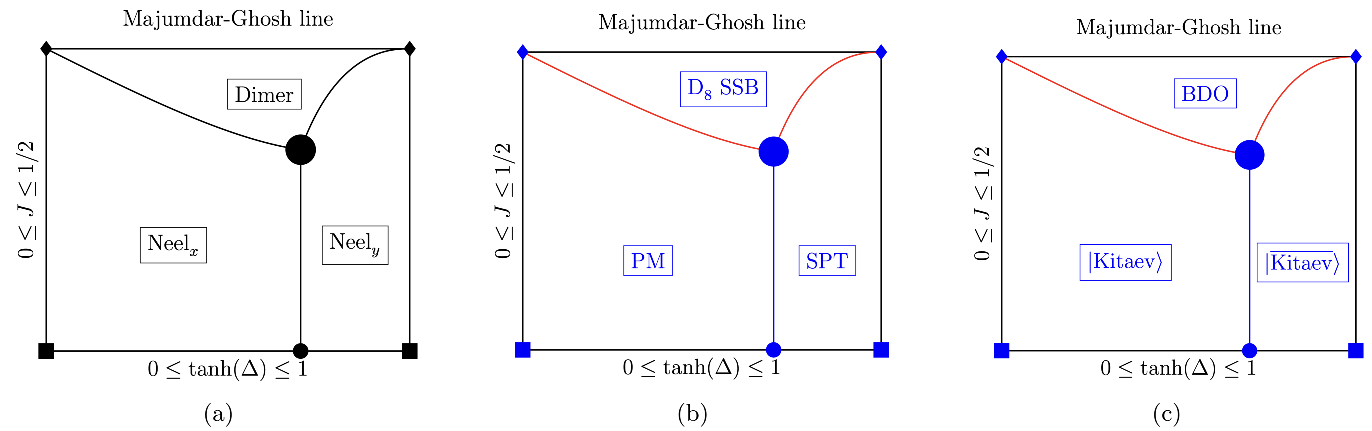

FIG. 2: XYZ phase diagram for (a) the original Hamiltonian, (b) after KW, and (c) after JW. (Images from [4])

degenerate ground states:

$$|1\rangle = |\downarrow, \rightarrow, \downarrow, \leftarrow\rangle^{\otimes N/2} \quad |2\rangle = |\downarrow, \leftarrow, \downarrow, \rightarrow\rangle^{\otimes N/2}$$
$$|3\rangle = |\rightarrow, \downarrow, \leftarrow, \downarrow\rangle^{\otimes N/2} \quad |4\rangle = |\leftarrow, \downarrow, \rightarrow, \downarrow\rangle^{\otimes N/2} \ . \quad (29)$$

Thus, the DQCPs between this phase and its neighbors have become standard SSB phase transitions after dualizing. This is indicated by the red phase boundaries in Fig. 2b, while the blue line indicates the SPT phase transition.

Finally, one may ask how we resolve the different degeneracies of the ground state before and after gauging. The answer boils down to the restriction of the Hilbert space to the even subsector of the gauged $\mathbb{Z}_2^z$ symmetry as mentioned before. After this restriction, the Neel phases have unique GHZ-like ground states, while the dual SSB phase has two GHZ-like ground states.

B. JW dual of XYZ model

Having elucidated the effect of the KW mapping on our phase diagram, let's now analyze the JW mapping. The new Hamiltonian is

$$H_f = J_1 \sum_j (\Delta_x i\beta_{j+1}\alpha_j + \Delta_y i\beta_j\alpha_{j+1}) + J_2 \sum_j (\Delta_x \beta_{j+1}\beta_{j+2}\alpha_j\alpha_{j+1} + \Delta_y \alpha_{j+1}\alpha_{j+2}\beta_j\beta_{j+1}) \ . \quad (30)$$

The new (mixed) symmetry group dual to $\mathbb{Z}_2^r \times \mathbb{Z}_2^x \times \mathbb{Z}_2^y$ is now given by $(\mathbb{Z}_4^{rF} \ltimes \mathbb{Z}_2^o \times \mathbb{Z}_2^e)/\mathbb{Z}_2^F$ where $\mathbb{Z}_4^{rF} \equiv \{r, r^2 = p_F, r^3, r^4 = e\}$ extends the reflection group by fermion parity.

The duals of the Neel$_x$ and Neel$_y$ phases are the two Kitaev chain topological phases. While these are often described as SPTs protected by fermion parity $\mathbb{Z}_2^F$, we typically don't allow terms which break this symmetry as they're very unphysical. In some sense, these phases therefore have intrinsic topological order. The fixed point wavefunctions of these phases are implicitly defined using

$$i\beta_{j+1}\alpha_j |\text{Kitaev}\rangle = -|\text{Kitaev}\rangle$$
$$i\beta_j\alpha_{j+1} |\overline{\text{Kitaev}}\rangle = -|\overline{\text{Kitaev}}\rangle \ , \quad (31)$$

up to boundary terms due to $f = 1$. The phases are topological because, after placing the system on open boundary conditions, two Majorana zero modes will appear at the ends of the chain. The degeneracy thus depends on the topology of space, and the transition between the two phases falls outside the standard Landau paradigm. Once again, the unique gapped ground state in each phase is allowed by the trivialized LSM anomaly.

The dual of the dimer phase is a bond density ordered (BDO) phase where neighboring Majorana operators will combine in pairs to form full complex fermions defined by

$$c_j = \frac{1}{\sqrt{2}}(\alpha_j + i\beta_j) \ . \quad (32)$$

Using this definition, we can write the fixed point wavefunctions of the BDO phase as

$$|\text{Bonding}_o\rangle \equiv \left[\prod_{j\in o} \frac{1}{\sqrt{2}}(c_j^\dagger + c_{j+1}^\dagger)\right] |0\rangle$$
$$|\text{Bonding}_e\rangle \equiv \left[\prod_{j\in e} \frac{1}{\sqrt{2}}(c_j^\dagger + c_{j+1}^\dagger)\right] |0\rangle \ . \quad (33)$$

Similar to the KW case, this phase breaks reflection symmetry and its phase boundaries are described by Landau's paradigm. However, the line between the two Kitaev phases is a topological phase transition. As before, we ensure the ground state degeneracies are respected by creating GHZ-like states in the Neel phases, but the 2-fold degeneracies in the dimer and BDO phases already match up.

V. CONCLUSION

We have analyzed the fate of Lieb-Schultz-Mattis anomalies after implementing the Kramers-Wannier and Jordan-Wigner transformations. In particular, we have seen how the LSM anomaly and the constraints it imposes on the ground state can be trivialized after such gauging dualities. However, this comes at a cost: the symmetries which remain after gauging mix into a more complicated group structure. Two such mixings we saw were the semidirect product of spatial and internal symmetries and the central extension of spatial symmetries by fermion parity. Another interpretation of this phenomenon is that spatially modulated symmetries, like that associated with conservation of dipole moment, appear after gauging with an LSM anomaly. We have seen the power of this method in explaining the connection between deconfined quantum critical transitions, (symmetry-protected) topological order, and spontaneous symmetry breaking in the quantum spin-1/2 XYZ model. Such dualities allow one phase diagram to be mapped to another, offering exact solutions to and analytic expressions for seemingly unrelated models.

For further reading, Refs. [2, 3] introduce and review the concept of LSM theorems. For a more exact treatment of the concepts in this paper, see [4]. Finally, for a look into how spatially modulated symmetries (like conservation of dipole moment) can affect models and their dynamics, see [5, 6].

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