Antiferromagnetic Heisenberg Model

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We introduce the parton mean field theory to the generalised antiferromagnetic ($J > 0$) Heisenberg model. We then show that there are underlying $SU(2)$ gauge redundancies regarding the hopping and pairing terms of the mean field Hamiltonian. From this knowledge, we construct a few inequivalent candidate spin liquid Hamiltonians and show that the π-flux phase and the chiral spin liquid are the two most likely candidates for the most likely J_2/J_1 values for the spin liquid phase at parameters where spin liquids are most likely.

I. INTRODUCTION

The Heisenberg model is one of the most well-studied model in condensed matter physics. A particular interest is given towards the 2D square Heisenberg model due to its relative simplicity compared to its higher dimensional analogues but still hosting many different magnetic phases of matter.

For the vanilla 2D Heisenberg model on a square lattice, the phase diagram remains simple: for $H = -J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j$, when $J < 0$ the system is ferromagnetic and when $J > 0$ it is antiferromagnetic [1]. From this model, we typically add the plaquette exchange term (commonly denoted as Q) or the next-nearest neighbour interactions. Considering the antiferromagnetic (Néel) state, there is a Néel to Valence-Bond Solid continuous phase transition at critical $Q = Q_c$ [2].

Another axis frequently added is the next-nearest hopping term; this is a reasonable assumption to make in that Coulomb interaction is long-range, and if we are going beyond tight-binding model by first order this will be the next term to consider. This direction is widely believed to host a spin liquid state, in a small range near $J_2/J_1 \approx 0.5$, i.e, at strong frustration before undergoing a further phase transition to the VBS and eventually columnar phases [3–6].

The outline of this paper is as follows. In Sec. II we introduce the Heisenberg model, then deconstruct this further into parton mean field theory. We then introduce various spin liquid phases in Sec. III and apply to the basic Heisenberg model. In Sec. IV we then further generalise this to the $J_1 - J_2$ model, and introduce chiral spin liquid states which can only start arising at this order. We will then compare various spin liquid mean field energy by values of J_2/J_1 by setting $J_2 = 1 - J_2$.

II. HEISENBERG MODEL AND PARTON MEAN FIELD THEORY

We start from the Hubbard model

$$H = -t\sum_{\langle ij\rangle\sigma} c^\dagger_{i\sigma}c_{i\sigma} + U\sum_i n_{i\uparrow}n_{i\downarrow} - \mu\sum_i n_i. \tag{1}$$

Considering the strong coupling limit ($U \gg t$) at half-filling ($\langle n\rangle = 1$), this reduces to the Heisenberg model

$$H = \frac{4t^2}{U}\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j = J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j. \tag{2}$$

which can be proved via the Schrieffer-Wolff transformation.

If we add the next-nearest neighbour terms as well, then this is the $J_1 - J_2$ model:

$$H_{J_1J_2} = J_1\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j + J_2\sum_{\langle\langle ij\rangle\rangle}\mathbf{S}_i\cdot\mathbf{S}_j. \tag{3}$$

We now introduce spinons $f_{i\alpha}$ obeying fermionic anti-commutation relations at site i of the lattice [7], with

$$\mathbf{S}_i = \frac{1}{2}f^\dagger_{i\alpha}\sigma_{\alpha\beta}f_{i\beta}. \tag{4}$$

where some algebra easily shows that $[S_i, S_j] = \frac{i}{2}\epsilon_{ijk}S_k$, giving the angular momentum relations. Going back to the simple Heisenberg model, this parton construction rewrites the Hamiltonian to

$$H = -\frac{1}{2}J_1\sum_{\langle ij\rangle}\left[f^\dagger_{i\alpha}f_{j\alpha}f^\dagger_{j\beta}f_{j\beta} + \left(\frac{1}{2}n_i - \frac{1}{4}n_in_j\right)\right]. \tag{5}$$

This construction naturally has an issue that we have now expanded the Hilbert space of the system from 2 to 4 states per site, and equivalence is reached only when there is one fermion per site: $f^\dagger_{i\alpha}f_{i\alpha} = 1$. This means that the second term in Eq. 5 gives a constant and therefore can be ignored.

When applying the typical mean-field construction, we therefore need to respect this and impose the constraint $\langle f^\dagger_{i\alpha}f_{i\alpha}\rangle = 1$. Defining $\langle f^\dagger_{i\alpha}f_{j\alpha}\rangle = \chi_{ij}$,

$$H_{MFT} = -\frac{1}{2}J_1\sum_{\langle ij\rangle}\left[(f^\dagger_{i\alpha}f_{j\alpha}\chi_{ji}) + \text{h.c.}) - |\chi_{ji}|^2\right] + \\ + \sum_i a_0(i)(f^\dagger_{i\alpha}f_{i\alpha} - 1). \tag{6}$$

for the Heisenberg model with the further restriction that the self-consistency relation $\langle f^\dagger_{i\alpha}f_{j\alpha}\rangle = \chi_{ij}$ having to be satisfied. The mean field construction immediately allows us to generalise the Heisenberg model to any spin-spin interaction:

$$H_{MFT} = \sum_{\langle ij \rangle} -\frac{1}{2} J_{ij} \left[(f^\dagger_{i\alpha} f_{j\alpha} \chi_{ji}) + \text{h.c.}) - |\chi_{ji}|^2 \right] + \\ + \sum_i a_0(i)(f^\dagger_{i\alpha} f_{i\alpha} - 1). \quad (7)$$

as we notice that the mean field construction depends on the indices of interacting bonds, not of the nearest-neighbour nature of the Heisenberg model. Now, $\langle ij \rangle$ indicates not the nearest neighbours, but all possible/designated pairwise interactions on ij. Our goal now is to find a solution which satisfies the self-consistency relation, and if there are many, selecting the solution which minimises $\langle \psi_{MFT} | H_{MFT} | \psi_{MFT} \rangle$.

III. CONSTRUCTION OF SPIN LIQUIDS AND GAUGE INVARIANCE

A. Gauge redundancies of the mean field Hamiltonian

The form of Eq. 7 immediately implies that there is a $U(1)$ symmetry inside, namely $f_{i\alpha} \to f_{i\alpha} e^{i\theta}$. Then, if χ_{ij} and $a_0(i)$ is a solution, there is a gauge redundancy where $\chi_{ij} e^{i(\theta_j - \theta_i)}$ and $a_0(i)$ is also a solution. Therefore, the next-order mean field theory must contain fluctuations of this $U(1)$ gauge field as well. Defining $\chi_{ij} = \bar{\chi}_{ij} e^{i a_{ij}}$,

$$H_{fluc} = \sum_{\langle ij \rangle} -\frac{1}{2} J_{ij} \left[(f^\dagger_{i\alpha} f_{j\alpha} e^{i a_{ji}} \bar{\chi}_{ij}) + \text{h.c.}) - |\bar{\chi}_{ij}|^2 \right] + \\ + \sum_i a_0(i)(f^\dagger_{i\alpha} f_{i\alpha} - 1). \quad (8)$$

Therefore, in this theory the excitations are described by spinons coupled to the $U(1)$ gauge field. In parton MFT, the results are qualitatively incorrect generally due to the Hilbert space changing when we split the electrons in half; to remedy this, the $U(1)$ gauge constraint must be implemented (i.e, glueing back the partons) to retrieve at least the qualitative results for the spin liquid Hamiltonian.

We can however, generalise this even further. So far, our mean field construction only involved on counting average hopping. However, the $U(1)$ gauge symmetry implies that Anderson-Higgs mechanism can come into play, indicating we can also consider the pairing term. From the parton deconstruction

$$H = \sum_{\langle ij \rangle} -\frac{1}{2} J_{ij} \left[f^\dagger_{i\alpha} f_{j\alpha} f^\dagger_{j\beta} f_{i\beta} + \frac{1}{2} f^\dagger_{i\alpha} f_{i\alpha} f^\dagger_{j\beta} f_{j\beta} \right]. \quad (9)$$

with the additional $\sum_i n_i$ term. Defining $\eta_{ij}\epsilon_{\alpha\beta} = -2\langle f_{i\alpha} f_{j\beta} \rangle$ and $\chi_{ij} = 2\langle f^\dagger_{i\alpha} f_{j\beta} \rangle$, we easily see $\eta_{ij} = \eta_{ji}; \chi_{ij} = \chi^\dagger_{ji}$. The mean field Hamiltonian is

$$H_{MFT} = \sum_{\langle ij \rangle} -\frac{3}{8} J_{ij} [(f^\dagger_{i\alpha} f_{j\alpha} \chi_{ji} + \eta_{ij} f^\dagger_{i\alpha} f^\dagger_{j\beta} \epsilon_{\alpha\beta}) + \text{h.c.}) \\ - |\chi_{ji}|^2 | - |\eta_{ji}|^2] + \sum_i [a^3_0 (f^\dagger_{i\alpha} f_{i\alpha} - 1) \\ + [(a^1_0 + i a^2_0)(f_{i\alpha} f_{i\beta} \epsilon_{\alpha\beta}) + \text{h.c.}]]. \quad (10)$$

The constraint now becomes $\langle f^\dagger_{i\alpha} f_{i\alpha} \rangle = 1$ and $\langle f_{i\alpha} f_{i\beta} \epsilon_{\alpha\beta} \rangle = 0$. This strongly implies an $SU(2)$ gauge invariance inside the mean field Hamiltonian [8, 9]. To see this explicitly, we first define the doublet ψ and matrix U such that

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f^\dagger_2 \end{pmatrix}, \; U = \begin{pmatrix} \chi^\dagger_{ij} & \eta_i j \\ \eta^\dagger_{ij} & -\chi_{ij} \end{pmatrix}. \quad (11)$$

we can rewrite the Eq. 10 in the general form

$$H_{MFT} = \sum_{\langle ij \rangle} \frac{3}{8} J_{ij} \left[\frac{1}{2} \text{Tr} \left(U^\dagger_{ij} U_{ij} \right) - (\psi^\dagger_i U_{ij} \psi_j + \text{h.c.}) \right] \\ + \sum_i a^l_0 \psi^\dagger_i \sigma^l \psi_i. \quad (12)$$

where σ are the Pauli matrices. We then can easily verify that the Hamiltonian is invariant under local $SU(2)$ transformation $\psi_i \to T_i \psi_i$ and $U_{ij} \to T_i U_{ij} T^\dagger_j$.

B. Variational approach to mean field theory

As a final piece to the puzzle, we now calculate the mean field energy from the mean field Hamiltonian we have derived. We employ the variational approach, where the physical wavefunction $P|\psi^{(U_{ij}}_{MFT}\rangle$ as the trial wavefunction where P is the projection operator (Note, for the physical wavefunction, we need projection so that each site has 1 fermions.). This however is hard, and we employ the approximate average energy minimisation where the projection is dropped:

$$E_{MFT} = \langle \psi_{mean} | \sum_{ij} J_{ij} S_i \cdot S_j | \psi_{mean} \rangle. \quad (13)$$

with $(U_{ij})_{\alpha\beta} = -2\langle \psi_{mean} | \psi_{i\alpha} \psi^\dagger_{j\beta} | \psi_{mean} \rangle$,

$$E_{MFT} = -\sum_{\langle ij \rangle} \frac{3}{16} J_{ij} \text{Tr} \left(U^\dagger_{ij} U_{ij} \right). \quad (14)$$

From here we minimise the energy by adjusting U_{ij}. Therefore, the full procedure is as follows. We first, using the mean field Hamiltonian, calculate the energy of the ground state provided they obey the self-consistency equation and single-occupancy condition. We can then find the optimal values of U_{ij} which will minimise the energy of the mean field ground state. We can then use the variational mean field energy to retrieve the optimal mean field energy of the system.

IV. MEAN-FIELD ANALYSIS OF NEXT-NEAREST NEIGHBOUR MODELS

Having introduced the symmetries and systematic construction of spin liquid hopping and pairing terms, in this section we now select the ansatze which correspond to the nearest and next-nearest level Heisenberg model and compare their mean field energies. We will restrict ourselves to simple cases where we need to consider two independent self-consistency relations (e.g., χ), and couplings only to next-nearest neighbours only. Both conditions significantly reduce the numbers of spin liquids we must consider for the full analysis.

A. Zero and π-Flux

We first start with the most basic models, which are the zero flux (also called the uniform RVB state) [7] and π flux [10] phase inside the plaquettes with nearest-neighbour coupling. These models are maximally $SU(2)$-symmetric and is denoted as SU2An0 and SU2Bn0 [11] in spin liquid classification.

For the zero flux (uniform RVB),

$$\chi_{i.i+x} = \chi,\ \chi_{i.i+y} = \chi. \tag{15}$$

The Hamiltonian is therefore given as

$$H_{MFT} = -\frac{J\chi}{2}\sum_i((f^\dagger_{i,\alpha}f_{i+x,\alpha} + f^\dagger_{i,\alpha}f_{i+y,\alpha} + \text{h.c.}) - 2\chi). \tag{16}$$

We have now reduced the problem into a tight-binding form. As usual, converting this into $q-$space,

$$H_{MFT} = \sum_i(-J\chi(\cos q_x + \cos q_y))f^\dagger_{q\alpha}f_{q\alpha} + JN\chi^2. \tag{17}$$

Which consists of one band. To satisfy the $\langle c^]_{i\alpha}\dagger c_{i\alpha}\rangle = 1$ condition, we must fill this band half-full as the total Brillouin zone can host $2N$ particles. Therefore,

$$E_{mean} = -2NJ\chi\int_{-\pi}^{\pi}\frac{dq_x}{2\pi}\int_{-\pi}^{\pi}\frac{dq_y}{2\pi}(\cos q_x + \cos q_y) + NJ\chi^2 \tag{18}$$

$$= -\frac{8}{\pi^2}NJ\chi + NJ\chi^2. \tag{19}$$

Minimising the free energy (variational approach) gives $\bar\chi = 4/\pi^2$, and the variational mean field energy is therefore

$$E_{var} = -\frac{3}{8}\sum_{\langle ij\rangle}J\bar\chi^2 = -\frac{3}{4}NJ\bar\chi^2 = -0.123NJ. \tag{20}$$

For the π-flux case,

$$\chi_{i.i+x} = \chi,\ \chi_{i.i+y} = (-)^{i_x}\chi. \tag{21}$$

We immediately see that the translation symmetry is broken, and there are two inequivalent points in the Brillouin zone — k_x is now ranged $-\pi/2$ to $\pi/2$.

$$H_{MFT} = \sum_i((f^\dagger_{i,\alpha}g_{i+x,\alpha} + g^\dagger_{i+x,\alpha}f_{i+2x,\alpha} + f^\dagger_{i,\alpha}f_{i+y,\alpha} - g^\dagger_{i+x,\alpha}g_{i+x+y,\alpha}) + \text{h.c}). \tag{22}$$

in $q-$space,

$$H_{MFT} = \sum_q(e^{iq_x}f^\dagger_{q\alpha}g_{q\alpha} + e^{iq_x}g^\dagger_{q\alpha}f_{q\alpha} + e^{iq_y}f^\dagger_{q\alpha}f_{q\alpha} - e^{iq_y}g^\dagger_{q\alpha}g_{q\alpha}) + \text{h.c}). \tag{23}$$

giving

$$H_{MFT} = \begin{pmatrix}\cos q_y & \cos q_x\\ \cos q_x & -\cos q_y\end{pmatrix}. \tag{24}$$

this easily gives

$$E_q = \pm\sqrt{\cos^2 q_x + \cos^2 q_y}, \tag{25}$$

and we fill the lower valence band of the system to satisfy the projection.

$$E_{mean} = -2NJ\chi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dq_x}{2\pi}\int_{-\pi}^{\pi}\frac{dq_y}{2\pi}\sqrt{\cos^2 q_x + \cos^2 q_y} + NJ\chi^2 \tag{26}$$

$$= -0.958NJ\chi + NJ\chi^2. \tag{27}$$

We see that $\bar\chi = 0.479$, and the variational mean field energy is

$$E_{mean} = -\frac{3}{4}NJ\chi^2 = -0.172NJ. \tag{28}$$

which means the π-flux phase has lower energy than the RVB state at all Js to mean-field order.

B. Incorporation of Next-Nearest Couplings

For a general $J_1 - J_2$ model,

$$H = J_1\sum_i(\mathbf{S}_i\cdot\mathbf{S}_{i+x} + \mathbf{S}_i\cdot\mathbf{S}_{i+y}) + J_2\sum_i(\mathbf{S}_i\cdot\mathbf{S}_{i+x+y} + \mathbf{S}_i\cdot\mathbf{S}_{i+x-y}) \tag{29}$$

on a square lattice with translation symmetry. In the mean field regime,

$$H_{MFT} = \sum_{\langle ij\rangle} -\frac{J_1}{2}((f^\dagger_{i\alpha}f_{j\alpha}\chi_{ji} + \text{h.c.}) - |\chi_{ji}|^2) + .\sum_{\langle\langle ij\rangle\rangle} -\frac{J_2}{2}((f^\dagger_{i\alpha}f_{j\alpha}\chi_{ji} + \text{h.c.}) - |\chi_{ji}|^2) + \sum_i a_0(i)(f^\dagger_{i\alpha}f_{i\alpha} - 1). \tag{30}$$

which allows us to consider us until next-nearest level hopping in the spin liquid phases.

The first obvious cases we can think will be the 0-flux and π-flux cases to next nearest neighbour, namely

$$\chi_{i,i+x+y} = \chi\sigma^3,\ \chi_{i,i+x-y} = \chi\sigma^3 \tag{31}$$

for the 0-flux and

$$\chi_{i,i+x+y} = \chi(\sigma^3 + \sigma^1),\ \chi_{i,i+x-y} = \chi(\sigma^3 - \sigma^1). \tag{32}$$

for the π-flux, where σ denotes Pauli matrices. We have denoted the π-flux in a slightly different way (commonly known as the maximal d-wave state). The two states are equivalent and is linked via a specific gauge transformation. The proof for this and general discussion on number of inequivalent spin liquids is outlined in Appendix A.

From the calculation above we can effectively read out the energy, which for the 0-flux is $E = \cos(q_x + q_y) + \cos(q_x - q_y)$ and for π-flux $E = \pm\sqrt{\cos^2(q_x + q_y) + \cos^2(q_x - q_y)}$. Performing the same integral we get the exact same result, just with J replaced with J_2 with no J_1 contribution.

The nontrivial case we can consider is the chiral spin liquid [12], whose ansatz is given as

$$\chi_{i,i+x} = i\chi_1, \chi_{i,i+y} = i\chi_1(-)^{i_x}, \tag{33}$$

$$\chi_{i,i+x+y} = -i\chi_2(-)^{i_x}, \chi_{i,i+x-y} = i\chi_2(-)^{i_x}. \tag{34}$$

This actually is the only nontrivial case when we restrict ourselves to two self-consistency relations maximum [13]. Classification of inequivalent spin liquids on a square lattice have been proven in a general setting using the concept of Invariant Gauge Group and Projective Symmetry Group [11], but we can understand this qualitatively. (insert) We notice that the translation symmetry is also broken, and the x-direction Brillouin zone shrinks by half as well. We can then perform the same calculation as we have done above for the Heisenberg model and convert it into Fourier space, whose result can be written as

$$H = -J_1\chi_1 \begin{pmatrix} \sin q_y & \sin q_x \\ \sin q_x & -\sin q_y \end{pmatrix} - J_2\chi_2 \begin{pmatrix} 0 & -i(\cos(q_x - q_y) + \cos(q_x + q_y)) \\ i(\cos(q_x - q_y) + \cos(q_x + q_y)) & 0 \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} -J_1\chi_1 \sin q_y & -J_1\chi_1 \sin q_x + iJ_2\chi_2(\cos(q_x - q_y) + \cos(q_x + q_y)) \\ -J_1\chi_1 \sin q_x - iJ_2\chi_2(\cos(q_x - q_y) + \cos(q_x + q_y)) & J_1\chi_1 \sin q_y \end{pmatrix}, \tag{36}$$

giving

$$E_\pm = \sqrt{(J_1\chi_1)^2(\sin^2 q_x + \sin^2 q_y) + (J_2\chi_2)^2(\cos(q_x + q_y) + \cos(q_x - q_y)^2}. \tag{37}$$

To solve this, given J_1 and J_2 we must minimise the energy on two different variables, each of them satisfying the self-consistency relations. Given the form of the integral this has to be done numerically:

We notice, that for $J_2/J_1 \approx 0.49$, the optimal value for χ_2 goes from negative value to positive value, meaning that the self-consistency relation is actually nontrivial this case and the solution is only valid when $J_2/J_1 > 0.49$. To see this, we shall plot on the axis J_2 while setting $J_2 = 1 - J_1$, to cover all ratios. Figure 1 shows the behaviour of χ_2 as we increase J_2.

The mean field energy by spin liquid phases to this order can now be plotted, shown in Figure 2.

There are two important observations for the final result. First, as the self consistency relation breaks down as χ_2 goes to zero, in this case the chiral spin liquid state reduces to the π-flux case to nearest coupling order. i.e, as J_2 is increased, we observe a continuous phase transition. Second, restricting ourselves to the $J_2/J_1 \approx 0.5$ region, we see that indeed the π-flux and the chiral spin liquid are the two likely spin liquid states, with $J_2/J_1 \approx 0.5$ lying almost right at the continuous phase transition point.

V. SUMMARY

We have introduced the Schwinger Boson decomposition of fermions, and derived a mean field Hamiltonian which can describe possible spin liquid effective theories. We then explored the symmetries involved and recast them in the general $SU(2)$ form and the gauge redundancies associated with it. From this formalism, we have explored the most likely spin liquid candidate Hamiltonian until next-nearest order and calculated their mean field energy. Results show that in the region where spin liquid is most likely to prevail, the π-flux phase and the chiral spin liquid are very close in energy with each other and are the most likely candidates. We note that in this region the chiral spin liquid almost breaks self consistency equation so further analysis is required.

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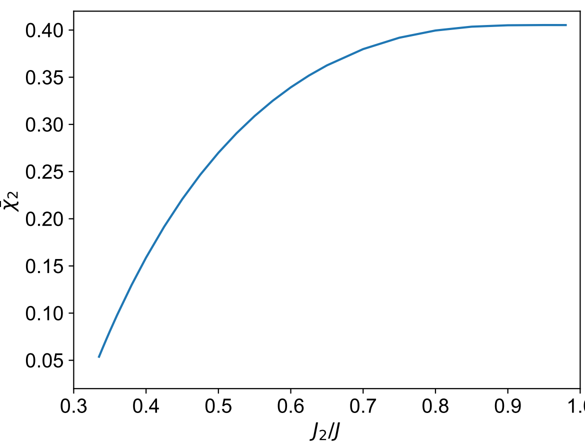

FIG. 1. Behaviour of χ_2 by changing J_2. We see that χ_2 does not obey self consistency relation when $J_2 \approx 0.33$, $J_1 = 1 - J_2 \approx 0.67$, or $J_2/J_1 \approx 0.49$.

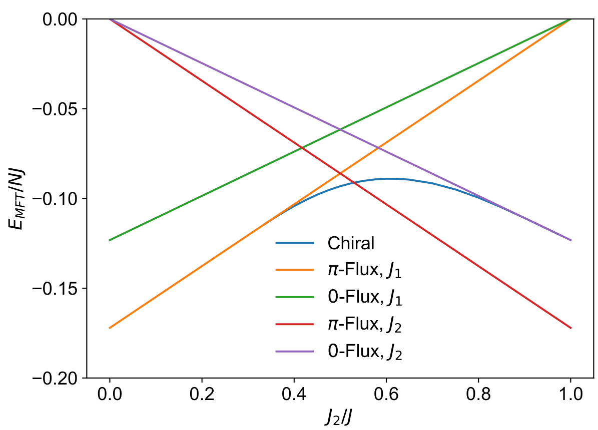

FIG. 2. Mean field energy for the π and 0-flux states mediated by the nearest and next nearest hoppings, alongside the chiral spin liquid which contains both terms simultaneously. We observe two phase transitions as we increase J_2.

Appendix A: Gauge transformation of different spin liquids

In this appendix, we show the most prominent example of the gauge transformation between $d-$wave state and the $\pi-$flux state. The π-flux hopping parameters are given as following:

$$\chi_{i.i+x} = -i\chi\sigma^0, \; \chi_{i.i+y} = -i\chi\sigma^0(-)^{i_x}. \tag{A1}$$

The d-wave spin liquid is given as

$$\chi_{i.i+x} = \chi(\sigma^3 + \sigma^1), \; \chi_{i.i+y} = \chi(\sigma^3 - \sigma^1). \tag{A2}$$

Let's consider the $SU(2)$ gauge transformation given by $W_i^\dagger = \left(i\frac{\sigma^3+\sigma^1}{\sqrt{2}}\right)^{i_x}\left(i\frac{\sigma^3-\sigma^1}{\sqrt{2}}\right)^{i_y}$. Then,

$$\chi_{ij} \to \left(-i\frac{\sigma^3-\sigma^1}{\sqrt{2}}\right)^{i_y}\left(-i\frac{\sigma^3+\sigma^1}{\sqrt{2}}\right)^{i_x}\chi_{ij}\left(i\frac{\sigma^3+\sigma^1}{\sqrt{2}}\right)^{j_x} \times \left(i\frac{\sigma^3-\sigma^1}{\sqrt{2}}\right)^{j_y}. \tag{A3}$$

Applying this transformation to $\chi_{i.i+x}$ (i.e, $j = i+1$),

$$\begin{aligned} \chi_{i.i+x} &= -i\chi\sigma^0 && \text{(A4)}\\ &\to -i\chi\sigma^0\left(i\frac{\sigma^3+\sigma^1}{\sqrt{2}}\right) && \text{(A5)}\\ &= \frac{\chi}{\sqrt{2}}(\sigma_3+\sigma_1), && \text{(A6)}\end{aligned}$$

and for $\chi_{i.i+y} = \chi(\sigma^3 - \sigma^1)$,

$$\begin{aligned} \chi_{i.i+y} &= -i\chi\sigma^0(-)^{i_x} && \text{(A7)}\\ &\to -i\chi\sigma^0(-)^{i_x}\left(i\frac{\sigma^3-\sigma^1}{\sqrt{2}}\right)(-)^{i_x} && \text{(A8)}\\ &= \frac{\chi}{\sqrt{2}}(\sigma_3-\sigma_1), && \text{(A9)}\end{aligned}$$

Therefore the two states are gauge-equivalent. Note, that while in this case we have rotated effectively $\pi/4$ in $SU(2)$ on σ_2 axis, the same principles can apply to all rotation axes and angles (which is $SO(3)$.). This effectively means that we can decompose the $SU(2)$ level degrees of freedom to simple $\mathbb{Z}_2$ degree of freedom, which are characterised by the 0- and π-flux. Hence, for one variable spin liquid, we have a complete picture, alongside two variable spin liquids are which are decoupled. Restricting ourselves to two-variable and to next-nearest order, the chiral spin liquid is effectively the only nontrivial spin liquid [11] from similar arguments, and other spin liquids which are competitive in general are gauge field fluctuations from the models given here which requires more variables, or require more than next-nearest level hopping.

[1] We are dealing with zero temperature cases, so in general the Mermin-Wagner theorem is inapplicable. In 1D, $J > 0$ actually leads to short-range order for even spins and quasi-long range order for odd spins.

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