Loop $O(n)$ model

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We discuss the loop $O(n)$ model, describing configurations of loops on a two-dimensional hexagonal lattice. The model contains as special cases many well-known models such as self-avoiding walks, Ising model, hard-core lattice gas, etc., and is also closely related to the spin $O(n)$ model. We also consider other representations as spin models with local interactions. We consider the conjectured phase diagram and transition, and then discuss some features which have been rigorously proven.

I. INTRODUCTION

The loop $O(n)$ model is a two-dimensional statistical model studying configurations of loops on a hexagonal lattice. Many specific cases of its hyper-parameters correspond to other models of interest, making this model of rather general interest.

More formally, let $\mathbb{H}$ denote the two-dimensional hexagonal lattice. A *loop configuration* is a spanning subgraph of $\mathbb{H}$ where each vertex has even degree. Any loop configuration consists of several *loops*, which are finite simple cycles in $\mathbb{H}$, as well as infinite *paths* and isolated vertices.

Let a *domain* $H \subset \mathbb{H}$ be a finite connected induced subgraph of $\mathbb{H}$ whose induced complement is also connected. In order to regularize the theory on the infinite lattice $\mathbb{H}$, in reality we will consider loop configurations with edges contained solely in H (so there are no infinite paths). Given such a loop configuration, let $L(\omega)$ be the number of loops in ω, and $o(\omega)$ the number of edges.

For any fixed positive real numbers n, x, we now assign a measure to $\mathsf{LoopConf}(H)$, which is the collection of all loops configurations on H. This *loop $O(n)$ measure* on $\mathsf{LoopConf}(H)$ is defined by

$$\mathbb{P}_{H,n,x}(\omega) = \frac{1}{Z_{H,n,x}} x^{o(\omega)} n^{L(\omega)}$$

where $Z_{H,n,x} = \sum_{\omega \in \mathsf{LoopConf}(H)} x^{o(\omega)} n^{L(\omega)}$ is the partition function.

II. CONJECTURED PHASE DIAGRAM AND CRITICALITY

The conjectured (n, x)-phase diagram of the loop $O(n)$ model, shown in Figure 1, is rather intricate. As we will discuss below, only parts of this conjectured diagram have actually been demonstrated rigorously.

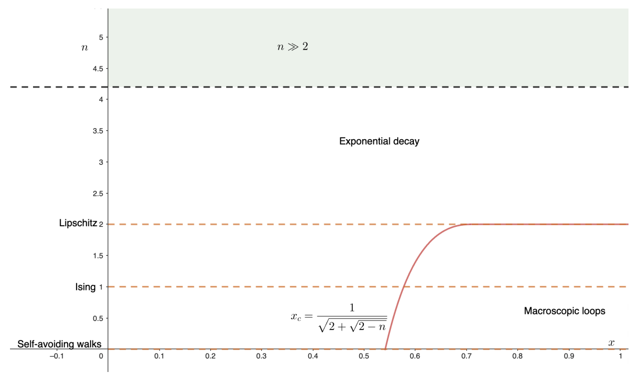

FIG. 1. The conjectured phase diagram of the loop $O(n)$ model, featuring a phase transition at $x_c = (\sqrt{2+\sqrt{2-n}})^{-1}$ for $n \in [0, 2]$. The exponential decay of loop lengths has only been rigorously proven for $n \gg 2$; most of the rest of the diagram is still conjectural. Special cases include self-avoiding walks and the Ising model.

A. Phase Transition

The main feature of this diagram is a conjectured phase transition, shown above in Figure 1. This phase transition describes the distribution of the length R of the largest loop surrounding the origin (a fixed vertex in $\mathcal{H}$). For each fixed valued of $n \in [0, 2]$, it is believed that there is a critical value of x,

$$x_c(n) = \frac{1}{\sqrt{2 - \sqrt{2-n}}}$$

which separates two regimes (at fixed n):

- For $x < x_c$, the model is *sub-critical.* This means that there exists some $c > 0$ such that for all $k \geq 1$ and all domains H, $\mathbb{P}(R > n) \leq \exp(-ck)$.
- For $x \geq x_c$, the model is *critical.* This means that $\mathbb{P}(R > n)$ scales as a power law instead; hence, the R is on the order of the radius of H.

For $n > 2$, it is believed that the model is always sub-critical.[1]

Furthermore, when $x \geq x_c$, the model has a mathematically well-defined conformally invariant scaling limit known as a *Conformal Loop Ensemble* (CLE) with parameter κ. There are two distinct regimes:

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- When $x = x_c(n)$, we pick the solution to $n = -2\cos\left(\frac{4\pi}{\kappa}\right)$ such that $\kappa \in \left[\frac{8}{3}, 4\right]$. In this regime, CLE curves almost surely never touch themselves.[2]

- When $x > x_c(n)$, we pick the solution to $n = -2\cos\left(\frac{4\pi}{\kappa}\right)$ such that $\kappa \in [4, 8]$. In this regime, CLE curves almost surely self-touch (but do not cross).

B. Known Results

When $n = 1$, we obtain the Ising model, which is well-studied. For $0 < x < 1$, the model corresponds to the ferromagnetic Ising model on a triangular lattice, with $x_c = \frac{1}{\sqrt{3}}$. The subcritical regime, the $CLE(3)$ scaling limit on $x = x_c$, and the existence of macroscopic loops for $x_c < x < 1$ are known. However, the convergence to $CLE(6)$ for $1/\sqrt{3} < x < 1$ remains open.

When $n = 0$, the model contains no loops at all. Hence, it is exactly the self-avoiding walk model. The critical point of this model is $x_c = \frac{1}{\sqrt{2+\sqrt{2}}}$, and it is known that the model obtains a dense phase for $x > x_c$.

Along the $n \in [1, 2]$, $x = x_c(n)$ line, it has been shown that the $O(n)$ model exhibits macroscopic loops.[1]

Recall that we expect the model to be sub-critical for all $n > 2$, for all values of $x > 0$. This has only been proven for sufficiently large $n \gg 2$ (in the sense that there exists some $n_0 > 2$ such that the model is subcritical for all $n > n_c$). In this regime, the so-called *hard-hexagon* transition also takes place, and will be discussed below.

For $x \ll 1$, it is simple to show that there is an exponential decay of loop lengths for all $n > 0$. More specifically:

Theorem 1 *For $n, x > 0$, consider the loop $O(n)$ model on some domain H. Then for any vertex $u \in V(H)$ and any positive integer k, we have*

$$\mathbb{P}_{H,n,x}(\exists \text{ loop of length } k \text{ surrounding } u) \le kn(2x)^k.$$

Here, a loop L is said to *surround* u if u is either in the interior or on the boundary.

Proof. Let a_k denote the number of simple *paths* of length k in $\mathbb{H}$ starting at some fixed vertex. It's obvious that $a_k \le 3 \cdot 2^{k-1}$ (there are 3 choices for the first move, and at most 2 choices for each of the remaining $k - 1$ moves). To bound the number of loops of length k surrounding u, consider drawing a "vertical path" starting from u, with k vertices. It's clear that any loop of length k surrounding u must intersect this vertical path. For any point on the vertical path, there are at most a_{k-1} loops through it of length k. Hence by union bound, the number of loops surrounding u is at most $ka_{k-1} \le k2^k$.

Finally, notice that for any loop L of length k, the probability that an element drawn from $\mathsf{LoopConf}(H)$ according to $\mathbb{P}_{H,n,x}$ is at most nx^k (for instance, one way to see this is that deleting L is an injective transformation which multiplies the Boltzmann probability by $1/(nx^k)$, and these results must sum to at most 1). Hence, by union bound across the $\le k2^k$ different loops surrounding u, we see that the desired probability is as most $(k2^k) \cdot (nx^k)$, as desired. □

As previously noted, this theorem implies that for $x \ll 1$ we have probabilities decaying exponentially with loop length.

III. SPIN $O(n)$ MODEL

The *spin* $O(n)$ model on a domain $H \subset \mathbb{H}$ is an ensemble of n-dimensional unit spins σ_v living on each vertex v of H, with local interactions between spins $\beta H = -\sum_{(u,v)\in E(H)} \langle \sigma_u, \sigma_v \rangle$. This model is related to the loop $O(n)$ model [1], with β corresponding to nx: they are believed to be in the same universality class, but this has not been rigorously proven. For small $\beta = nx$, one can suggestively compute that the partition functions of the two models are approximately equal:

$$\begin{aligned}
Z^{\text{spin}}_{H,n,\beta} &= \left(\int_{S^{n-1}} d\Omega\right)^{|V(H)|} \prod_{(u,v)\in E(H)} \exp(\beta\langle \sigma_u, \sigma_v\rangle) \\
&\overset{(1)}{\approx} \left(\int_{S^{n-1}} d\Omega\right)^{|V(H)|} \prod_{(u,v)\in E(H)} (1 + \beta\langle \sigma_u, \sigma_v\rangle) \\
&\overset{(2)}{=} \sum_{\omega \in \mathsf{LoopConf}(H)} \left(\frac{\beta}{n}\right)^{o(\omega)} \left(\int_{S^{n-1}} d\Omega\right)^{|V(H)|} \\
&\qquad \prod_{(u,v)\in E(\omega)} \langle \sqrt{n}\sigma_u, \sqrt{n}\sigma_v\rangle \\
&\overset{(3)}{=} \sum_{\omega \in \mathsf{LoopConf}(H)} \left(\frac{\beta}{n}\right)^{o(\omega)} n^{L(\omega)}.
\end{aligned}$$

Here step (1) is the approximation $e^t \approx 1 + t$, step (2) follows from expanding the product into a sum over $\omega \in \mathsf{LoopConf}(H)$, and step (3) is the fact that for any loop configuration ω we have

$$\left(\int_{S^{n-1}} d\Omega\right)^{|V|} \prod_{(u,v)\in E(\omega)} \langle \sqrt{n}\sigma_u, \sqrt{n}\sigma_v\rangle = \begin{cases} n & \text{if } E \text{ is a loop} \\ 0 & \text{otherwise} \end{cases}.$$

This approximation is not rigorously justified for any positive x whenever $n > 1$, but nevertheless, it lends strong support to the belief that the two models are in the same universality class.

[1] note the loop $O(n)$ model is defined for any positive real n, while the $O(n)$ spin model is only defined for positive integer n. When relating the two, we're taking n to be a positive integer.

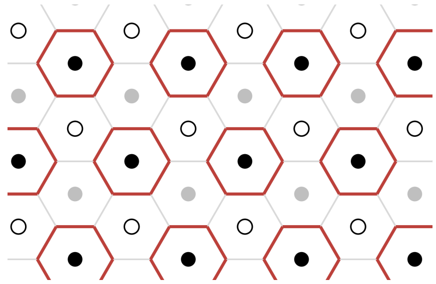

FIG. 2. The centers of $\mathbb{H}$ can be three-colored as shown in black, white, and gray. There is a ground state of the hard hexagon model corresponding to the black hexagons; in the loop $O(n)$ model this corresponds to the red edges. There are clearly two other similar ground states.

IV. EQUIVALENT MODELS

A. Hard-Hexagon Model

Suppose we take the limit $n \to \infty$ as $nx^6 = \lambda$ is held fixed, known as the hard-hexagon model. In this case, loops with length ℓ larger than 6 become suppressed, since they contribute Boltzmann weight $nx^\ell = \lambda x^{\ell-6} \to 0$, so such loops no longer appear in the limit. We are left solely with configurations containing *hexagons*, which are loops of length 6. Each hexagon contributes a factor of $n^1 x^6 = \lambda$ to the weight; hence this hard-hexagon model is exactly the hard-core lattice model on the dual triangular lattice $\mathbb{T}$ with fugacity λ.

Observe that there are three lowest-energy states in the hard-hexagon model, as shown in Figure 2.

As λ crosses a critical fugacity $\lambda_c = \frac{1}{2}\left(11 + 5\sqrt{5}\right)$, it is known that the model undergoes a fluid-solid phase transition. This means that the model transforms from a homogeneous phase where each sublattice is occupied with equal frequency, to an ordered symmetry-broken phase where one of the three sub-lattices is occupied more.

V. ALTERNATIVE REPRESENTATIONS AS SPIN MODELS WITH LOCAL INTERACTIONS

The relationship to the spin $O(n)$ model discussed previously was approximate. We now present a completely different approach, which is exact.

A. Discrete Spins and Domain Walls

The spins will be discrete, lying in a countable set S. Let Φ be the set of $\varphi \in S^{\mathbb{T}}$ such that, among any three mutually adjacent hexagons $y, z, w \in \mathbb{T}$, at least two of $\varphi(y), \varphi(z), \varphi(w)$ are the same.

Specifically, such φ have at most one unbounded component of equal spins, with all other components finite, and possibly containing other smaller non-contacting components of other spins inside of themselves. Thus, it is completely natural to define the *domain walls* of a configuration φ as the collection of edges $e \in E(\mathcal{H})$ (which are dual to edges in $\mathbb{T}$) marking the boundaries between hexagons $y, z \in \mathbb{T}$ with different spins $\varphi(y) \neq \varphi(z)$. Of course, this naturally yields a loop configuration ω_φ.

To add in boundary conditions, we can fix some domain $H \in \mathcal{H}$ and some ambient infinite spin s_0 which we imagine fills the border. To this end, let $\Phi(H)$ be the set of $\varphi \in \Phi$ such that $\varphi(z) = s_0$ for all hexagons $z \in \mathbb{T}$ which are not contained in H; then $\omega_\varphi \in \mathsf{LoopConf}\,H$.

Now, consider a non-zero symmetric interaction $h : S^3 \to [0, \infty)$, such that $h = h \circ \tau$ for any permutation τ. Then, we place the measure

$$\frac{1}{Z} \prod_{y,z,w} h\left(\varphi(y), \varphi(z), \varphi(w)\right)$$

on the configurations, where the summation is over triples of mutually adjacent hexagons in $\mathbb{T}$ which intersect H. Note that in the case where S is countable, care must be taken to ensure Z can be made finite.

Of course, because we are only considering $\varphi \in \Phi$, we only care about the values of $h_a = h(a,a,a)$ and $h_{a,b} = h(a,b,b)$.

Thus, a choice of S and h yields a measure which can be pushed forward through $\varphi \to \omega_\phi$ to a probability distribution on $\mathsf{LoopConf}\,H$. We will now show how to choose S, h to replicate $\mathbb{P}_{H,n,x}$.

B. Replicating $\mathbb{P}_{H,n,x}$

Following our nose, allow $h_a \equiv 1$ for all a, such that the measure is weighted solely by the domain walls.

For simplicity, we take S to be finite. Our construction will use a simple graph G on S. Its adjacency matrix A is a real symmetric matrix, hence by the Perron-Frobenius theorem, there exists some maximal eigenvalue λ and eigenvector $A\psi = \lambda\psi$. We thus set:

- $h_{a,b} = x\left(\frac{\psi_a}{\psi_b}\right)^{1/6}$ for $(a,b) \in E(G)$.
- $h_{a,b} = 0$ otherwise.

As a side note, this distribution on $\Phi(H)$ is supported on the collection of *Lipschitz* spin configurations on G; if $y, z \in \mathbb{T}$ are adjacent, then $\varphi(y), \varphi(z)$ are at most a distance of 1 apart on graph G.

Now, if ψ is randomly sampled according to this h, then we claim ω_φ is distributed as $\mathbb{P}_{H,\lambda,x}$. To show this, for each fixed $\omega \in \mathsf{LoopConf}(H)$, we must sum the weights over the pre-image $\{\varphi : \omega_\varphi = \omega\}$.

Equivalently, we must perform a summation over all possible ways to color each contiguous block of spins. At this juncture, observe that the topology of a loop configuration naturally arranges the collection of contiguous blocks into a tree structure called the *block tree*, where

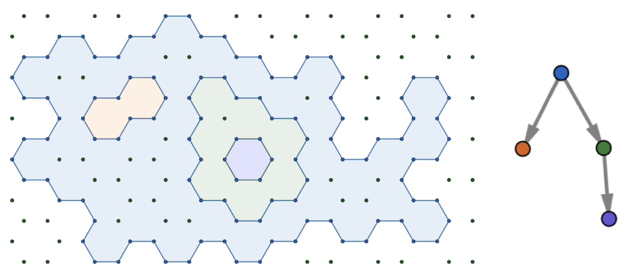

FIG. 3. An example of the block tree corresponding to a configuration.

blocks are the ancestor of all blocks contained within it. See Figure 3 for an example of a block tree.

Thus, the claim is that for any *fixed* color s_0 of the root node of the block tree, the total weight of the entire tree is equal to the product of $x^{|\ell|}\lambda$. This will be shown inductively. When the tree contains one node, we have a vacuous base case.

Now, if the root node is fixed to the color a, then we observe that each of the children of this root node yield independent weights, thus we can simply multiply the weight contributions of each subtree. For some particular child, we simply sum over all possibilities b for its color – the loop contributes $\sum_{b\in S} h^m_{b,a}h^{m'}_{a,b}$, where m and m' are the number of vertices of ℓ which touch an edge in the exterior and interior sides of ℓ, respectively.

Of course, by adding together all of the angles turned, we obtain that $m = m' + 6$. Thus,

$$\sum_{b\in S} h^m_{b,a}h^{m'}_{a,b} = x^{m+m'} \sum_{b\in S:\{a,b\}\in E(G)} \left(\frac{\psi_b}{\psi_a}\right)^{\frac{m-m'}{6}}$$

$$= x^\ell \frac{(A\psi)_a}{\psi_a} = x^\ell\lambda.$$

This concludes the proof. We obtain:

Theorem 2 *For all λ such that there exists a finite graph G whose adjacency matrix has maximum eigenvalue λ, there exists a discrete-spin model whose partition function corresponds to the $n = \lambda$ $O(n)$ loop model.*

C. Limitations and Special cases

Unfortunately, not all real numbers n are obtainable in this fashion, as the collection of finite graphs is countable while the real numbers are not. The collection of possible n does include many cases of interest though. It is in fact known that the set of possible n in $(0, 2)$ are the maximal eigenvalues of the Dynkin diagrams of types A,D,E.

On the other hand, one can extend the above construction to certain countable S and infinite G, such as if G is locally finite. Results such as the Krein–Rutman theorem allow us to prove existence of eigenvectors. The salient point is that if there exists some non-zero $\psi \in \mathbb{R}^S_{\geq 0}$ such that $\lambda\psi_a = \sum_{b:\{a,b\}\in E(G)} \psi_b$ for some $\lambda > 0$, then the above arguments yield a countable discrete-spin model reproducing $\mathbb{P}_{H,\lambda,x}$.

As a special case of the above construction, if n is a positive integer, then we can simply let $G = T_n$ be the (infinite) n-regular tree. Then, the set of 1-Lipschitz functions from $\mathbb{T} \to T_n$ are naturally identified with the support on Φ. Such functions to T_n are called height functions. In the case of the Ising model $n = 1$, $G = \{+, -\}$ and the correspondence is obvious. In the case of $n = 2$, we have the so-called *restricted Solid-On-Solid model.*

As another special case, if $n = \sqrt{q}$ we might consider the *star graph* on $q + 1$ vertices $\{0, \ldots, q\}$ with 0 at the center. The eigenvector is $(\sqrt{q}, 1, \ldots, 1)$. This naturally corresponds to the *dilute Potts model*, where 0 represents a vacancy and the positive integers represent Potts spins.

VI. LARGE n

In this section we will provide a sketch of the proof of the result that for large n, long loops are exponentially unlikely to occur [2]. More precisely, say a loop L *surrounds* a vertex v if v is either on L or contained in its interior. Also, let ω^0_g denote the "ground state" of the model consisting of maximally packed hexagons (note there are two other similar ground states ω^1_g and ω^2_g as shown in Figure 2; simply label these arbitrarily), and say a domain $H \subset \mathbb{H}$ is of *type 0* is every edge in ω^0_g has either neither or both of its endpoints in H. This is just a boundary-niceness condition on H in order to state the following theorem:

Theorem 3 *There exists positive constants n_0, c such that the following is true: take any type 0 domain H and weights $n \geq n_0, x > 0$. Then for any $u \in V(H)$ and any integer $k > 6$ we have*

$$\mathbb{P}_{H,n,x}\left(\exists \text{ loop of length } k \text{ surrounding } u\right) \leq n^{-ck}.$$

Sketch of Proof. It's easiest to separate into two cases, depending on the relative sizes of n and x.

Case 1: Suppose that $nx^6 < n^{1/50}$, or equivalently $x \leq n^{-49/300}$. Since we're taking n sufficiently large, we can assume that e.g. $2x \leq n^{-4/25}$. Also note that for sufficiently large n, we have $t \leq n^{t/120}$ for any $t \geq 0$. So by Theorem 1, we have the desired probability is (using that $k > 6$)

$$\mathbb{P} \leq kn(2x)^k \leq kn^{1-4k/25} \leq kn^{-k/60} \leq n^{-k/120}.$$

[2] For the proof with all the details written out, see e.g. [3]; it is too long to be appropriate for this paper.

Case 2: Now suppose $nx^6 \geq n^{1/50}$. Consider some loop L of length k surrounding u; it is not hard to show by some omitted combinatorial arguments that there exists a constant D such that the probability that L exists is at most

$$\mathbb{P}_\omega(L \subset \omega) \leq \sum_{\ell=1}^{\infty} D^\ell \left(\min(nx^6, n)\right)^{-\max(k,\ell)/15}.$$

Since $nx^6 \geq n^{1/50}$ and n is large, we can assume $\min(nx^6, n) > D^{15}$, and hence the above is at most $(C\min(nx^6,n))^{-k/15} \leq (Cnx^6)^{-k/15} \leq (Cn^{1/50})^{-k/15}$ for some constant C. Now recall from the proof of Theorem 1 that there are at most $k2^k$ loops surrounding u. So by union bound, the desired probability is at most

$$(k2^k) \cdot (Cn^{1/50})^{-k/15} \leq n^{-k/800}.$$

for sufficiently large n. □

VII. CONCLUSION

In this paper we discussed various aspects of the loop $O(n)$ model on the hexagonal lattice $\mathbb{H}$. We showed its relationship with the spin $O(n)$ model and exact representations as local spin models, as well as how it contains several well-known models as special cases. We discussed the conjectured phase diagram and transition, which contain very interesting (conjectured) features such as a critical line along $x_c = \frac{1}{\sqrt{2+\sqrt{2-n}}}$ for $n \in [0,2]$; we presented proofs and sketches of some the few rigorously known results.

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