Entanglement renormalization and topological phases

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Entanglement renormalization group techniques for tensor network models enable efficient local representations and the simulations of a broad class of quantum state. Furthermore, they provide a tractable framework for analyzing the behavior of quantum information and entanglement dynamics in these systems. In this note we review and provide intuition for the multiscale entanglement renormalization ansatz (MERA), and discuss its application to classifying topological phases of matter. We then use MERA to reproduce recent results on the universality of the topological entanglement entropy.

I. INTRODUCTION

Tensor networks can be understood as an extension of classical techniques for solving isotropic spin lattice models under the relaxation of translation invariance [1], and enable efficient classical representations of many-body states which are generated by local dynamics. These methods have therefore gained traction for their applications in practical numerical simulation, as well as for providing tractable models of quantum information dynamics in many-body physics [2].

The simplest example of a quantum state which can be represented as a tensor network is the *matrix product state*, which represents a state with finite correlation length and loosely speaking corresponds to the ground state of a local gapped Hamiltonian; this relationship is known to be exact in 1D [3, 4]. On the other hand, it is often useful to consider critical models corresponding to gapless ground states, which exhibit qualitatively different behavior. For instance, while the matrix product state structure enforces a strict exponential decay of correlators, gapless states are generically observed to result in power law decay [2].

The multiscale entanglement renormalization ansatz (MERA) addresses these issues by extending decimation-inspired techniques, in particular the density matrix renormalization group (DMRG) algorithm, in a process known as *entanglement renormalization* to describe a broader class of states [5]. In addition, these methods scale efficiently in higher dimensions, providing a useful analytic and computational framework.

Notably, these methods also provide an effective model for quantum phases of matter which can be distinguished by their entanglement structure. For instance, while matrix product states must obey area law entanglement scaling, MERA generically exhibits a logarithmic divergence, reproducing the properties of many gapless states [2]. In addition, topological phases of matter, which do not correspond to a traditional order parameter, are characterized by certain patterns of long-range entanglement and ground state degeneracies [6]. MERA thus enables the efficient simulation of models of these phases, offering insights into methods of classification [7, 8].

In this note we provide an overview of tensor network methods, first building intuition by reviewing matrix product states and the classification of gapped 1D phases in Section II. We then introduce the entanglement renormalization coarse-graining procedure in Section III which allows us the construct the MERA framework. Then, in Section IV we consider the application of MERA to describe topological phases, with a focus on reproducing recent results regarding the universality of the topological entanglement entropy. Finally, in Section V we discuss

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connections to broader issues and potential for future work in this direction.

II. PRELIMINARIES

A. Matrix product states

We begin by reviewing the 1D matrix product state formalism. For any state $|\psi\rangle$ on a multipartite system of qudits with local dimension q and subsystem A, recall the Schmidt decomposition guarantees a representation of $|\psi\rangle$ of the form

$$|\psi\rangle = \sum_i a_i |i\rangle_A \otimes |i\rangle_{\overline{A}}, \tag{1}$$

where the sets $\{|i\rangle_A\}$ and $\{|i\rangle_{\overline{A}}\}$ form orthonormal bases on A and $\overline{A}$ respectively, and $\sum |c_i|^2 = 1$. This is equivalent to the statement that the reduced states on A and $\overline{A}$ have the same spectrum and entropy. For any 1D system with linear geometry and bounded bipartite Schmidt rank, it is possible to decompose the state via a matrix product state construction. In particular, suppose there exists constant b such that for any $m \in [n]$, $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_{i=1}^{b} a_i |i\rangle_{[m]} |i\rangle_{[m+1,n]}. \tag{2}$$

Then we can write $|\psi\rangle$ in a matrix product state form as a product of tensors

$$|\psi\rangle = A^{(1)}_{i_1,q'_1} A^{(2)}_{i_1,i_2,q'_2} \dots A^{(n)}_{i_{n-1},q'_n} \tag{3}$$

$$= \left| A^{(1)} A^{(2)} \dots A^{(n)} \right\rangle, \tag{4}$$

where except for the first and last site each tensor is of dimension qb^2. This corresponds to a chain of n tensors connected by bonds representing the contracted virtual indices, and dangling legs representing the physical indices as in Figure 1.

Notably, since the total bond dimension is upper bounded by the product of the bond dimensions of the corresponding contracted tensors, applying a fixed-depth local circuit on the system can increase the bond dimension by at most a constant factor. Thus states which are prepared via a series of local interactions can be efficiently simulated in this manner, with complexity scaling as $O(nq)$. In addition, we remark that for a sufficiently generic system, i.e. with chaotic time-dependent dynamics the typical bond dimension is equal to the maximum.

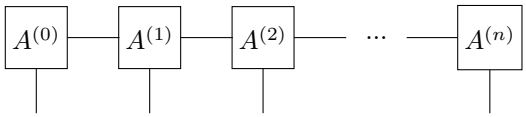


FIG. 1: Matrix product state representation of a state on n qudits

We can apply a transfer matrix method to analyze the behavior of correlation functions. Consider an operator $\mathcal{O}$ and the two point correlator

$$\langle\psi| \mathcal{O}_0 \mathcal{O}_r |\psi\rangle. \tag{5}$$

We can then define the corresponding transfer matrix at the site 0:

$$T^{(0)}_{\mathcal{O}} = \sum_{i,j=1}^{b} \mathcal{O}_{ij} A^{(0)}_i \otimes A^{(0)}_j. \tag{6}$$

Furthermore, we denote by $T^{(k)}$ the transfer matrix associated to site k, with $\mathcal{O} = \mathbb{1}$. Then in the thermodynamic limit the two point correlation function can be written as

$$\mathrm{Tr}\left[T^{\infty} T^{(0)}_{\mathcal{O}} T^{(1)} \dots T^{(r-1)} T^{(r)}_{\mathcal{O}} T^{\infty}\right]. \tag{7}$$

It is clear that this quantity scales as the product of the largest eigenvalues of the $A^{(k)}$, and therefore is either constant for special cases or decays exponentially in r.

B. Density matrix renormalization group

We now describe the DMRG algorithm, which is the basis for a set of variational numerical methods for obtaining the low-energy solutions of a system. While this technique was developed independently of other tensor network methods, it can be naturally understood within

the matrix product state framework [9, 10]. Recall the variational problem of finding the ground state is given by

$$\arg\min_{A^{(k)}} \frac{\left\langle \psi(A^{(k)}) \right| H \left| \psi(A^{(k)}) \right\rangle}{\left\langle \psi(A^{(k)}) | \psi(A^{(k)}) \right\rangle}. \tag{8}$$

For a given k, we define the tensors $\mathcal{H}_k, \mathcal{I}_k$ such that

$$\left\langle \psi(A^{(k)}) \right| H \left| \psi(A^{(k)}) \right\rangle = \left\langle A^{(k)} \right| \mathcal{H}_k \left| A^{(k)} \right\rangle, \tag{9}$$

$$\left\langle \psi(A^{(k)}) | \psi(A^{(k)}) \right\rangle = \left\langle A^{(k)} \right| \mathcal{I}_k \left| A^{(k)} \right\rangle. \tag{10}$$

Note that for physical $|\psi\rangle$, there exists a representation of ψ such that $\mathcal{I}_k$ is simply the identity. It is thus possible to obtain an approximate local solution by starting with a physical ansatz which captures a truncated spectrum of the Hamiltonian and applying a greedy approach which iterates over k, assigning

$$A^{(k)} \leftarrow \arg\min_{A^{(k)}} \left\langle A^{(k)} \right| \mathcal{H}_k \left| A^{(k)} \right\rangle. \tag{11}$$

Furthermore, by combining adjacent tensors to obtain coarse-grained sites before minimizing, then using the singular value decomposition (SVD) to split apart the resulting tensor, it is possible to gradually increase the bond dimension and resolution of the approximation.

C. Classification of gapped 1D phases

Recall that classical phase transitions correspond to nonanalytic behavior of the free energy density in the thermodynamic limit, producing divergence in the correlation length. In the quantum case, we are typically interested in the behavior of low-energy states for a given class of Hamiltonians, whose measurement statistics are described by the entanglement dynamics of the system and may also exhibit a diverging correlation length in certain cases.

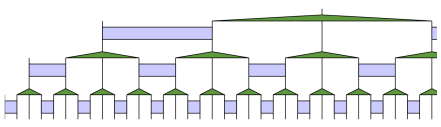
In 1D, tensor network methods enable the complete classification of symmetry-protected-topological (SPT) phases in gapped quantum systems [11], i.e. periodic systems which feature a finite energy gap between the ground state and the lowest excitation energy in the thermodynamic limit. Note that the transition corresponds physically to a point in phase space with a gapless spectrum.

We distinguish local SPT phases from plain topological phases, which are defined as an equivalence class over the region of the Hilbert space which can be accessed by smoothly deforming the Hamiltonian for a fixed time t, or by applying a fixed-depth quantum circuit. In the absence of additional structure, all matrix product states are thus in the same phase as the product state $|0\rangle^{\otimes n}$, and are therefore in the "trivial" phase.

Moreover, within this framework, protected symmetries completely decompose the space of allowed unitary transformations into commuting blocks, and can therefore be absorbed into the virtual bonds [2]. The symmetries of the $A^{(k)}$ tensors therefore completely determine the SPT phases in 1D [12].

III. MULTISCALE ENTANGLEMENT RENORMALIZATION ANSATZ

While matrix product states and their counterparts in higher-dimensions provide a simple model for many physical systems of interest, they are limited by their ability to model only strictly local correlations. MERA addresses this issue by parameterizing different length scales in the system via entanglement renormalization, in which successively coarse-grained layers are applied on top of a D-dimensional state to produce a $D+1$-dimensional tensor network. We can visualize this for the 1D case as in [2]:

$|\psi\rangle =$  , (12)

where rectangles U and triangles V denote unitary and isometric transformations respectively:

$$U^\dagger U = \mathbb{1}, \quad V^\dagger V = \mathbb{1}, \tag{13}$$

where reading top-down, the input of each transformation has the same virtual dimension b, and each degree of freedom which is coarse-grained out can be interpreted as inputting the product state $|0\rangle$ to produce the state $|\psi\rangle$ on n qudits at the bottom.

Alternatively, given $|\psi\rangle$, the unitary transformations can be interpreted bottom-up as disentangling operations necessary for to produce tensors with fixed bond dimension in the decimation process [13]. Moreover, as this dimension is a constant which does not scale with the size of the system, these can be efficiently computed via SVD and store information about correlations between different coarse-grained sites.

Recall that matrix product states feature a constant number of virtual bonds at each point in the boundary of a region A, and therefore area law entanglement

$$S_A \sim b[A]. \tag{14}$$

On the other hand, the tree structure of MERA requires $O(\log|A|)$ bonds between any connected subregion and the rest of the system, resulting in a logarithmic divergence

$$S_A \sim b \log|A|. \tag{15}$$

Furthermore, it is possible to obtain power law scaling for two-point correlation functions by considering the recursion depth needed to entangle two sites is given by $\Delta \sim \log r$, and using the fact that $\mathcal{O}_k$ commutes with any unitary or isometry operations from which the site k on the bottom layer is not fed into. The behavior of correlators is therefore controlled by the typical magnitude of the largest eigenvalue λ via

$$\langle\psi|\,\mathcal{O}_0\mathcal{O}_r\,|\psi\rangle \sim \lambda^{\log r/c} = r^{c\log\lambda}. \tag{16}$$

MERA thus reproduces the qualitatively correct scaling for the critical gapless system.

IV. TOPOLOGICAL PHASES IN MERA

We now turn to the application of MERA to describe states with non-trivial topological order, which necessarily exhibit long-range entanglement. Recall that topological phases can be defined as an equivalence class with respect to circuits of constant depth corresponding to the natural time scale t. Then a state is in the trivial phase if it is possible to reach the product state fixed point under the renormalization group action sending

$$l \to l/c^{\Delta(t)}. \tag{17}$$

Otherwise, we observe that the remaining entangled degrees of freedom in the top tensor are precisely those that are topological, representing a type of "non-trivial fixed point". In the case of the translation-invariant states such as the toric code, this statement is exact [13, 14].

A. Topological entanglement entropy

It is often useful to discuss the presence of topological order in arbitrary states, without reference to an explicit Hamiltonian. One proposal for the characterization of generic topological phases is the topological entanglement entropy, which describes a constant correction to the area law scaling expected in local gapped systems [15, 16]:

$$S_A \sim b[A] - \gamma + \ldots, \tag{18}$$

where $\gamma > 0$ is a constant identified with the topological entanglement entropy and the $\ldots$ terms vanish in the thermodynamic limit.

While computing the exact value relies on knowledge about the microscopic dynamics of the system, it is possible to approximate γ by explicitly computing the entropy of finite subregions and canceling out contributions from the boundaries. For example, consider the annulus construction illustrated in Figure 2, for which the TEE is given in terms of the three subregions A, B, C:

$$2\gamma = S_{AB} + S_{BC} - S_B - S_{ABC} \tag{19}$$

$$= I(A:C|B). \tag{20}$$

This definition yields the same value as Eq. 18

in the limit as the size of the regions goes to infinity. We remark that the value computed us-

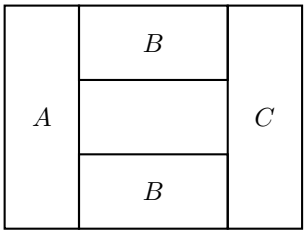


FIG. 2: Illustration of annulus construction

ing this construction is in general not a topological invariant, as it can change spuriously under the application of a fixed-depth circuit. Furthermore, a positive value may be obtained for some states in the trivial phase [17, 18]. In the following section we discuss a recent result regarding the universality of γ and produce a different argument supporting this conclusion using MERA.

B. Universality in MERA

It was recently shown in [19] using Markov state techniques and the strong subadditivity of mutual information that for a broad class of models exhibiting topological order, the topological entanglement entropy cannot decrease under the application of a local circuit which is shallow compared to the natural length scale of the system. We provide an alternative argument which reproduces this result for reference states which can be described by MERA.

Recall that for a given $|\psi\rangle$ and fixed Δ_0 with

$$\Delta_0 \lesssim \Delta(t), \tag{21}$$

we have that under the entanglement renormalization group action, we obtain a coarse-grained representation

$$|\psi\rangle = \mathcal{V}^\dagger(|\tilde{\psi}\rangle \otimes |0\rangle), \tag{22}$$

where $\mathcal{V}$ denotes the composition of the unitary and isometric transformations in the process and $|\tilde{\psi}\rangle$ by definition contains the same topological information as $|\psi\rangle$. The topological phase is therefore generated by the choice of $\mathcal{V}$.

Now let γ_0 denote the topological entanglement entropy of $|\tilde{\psi}\rangle$, as defined in Eq. 19, and γ' denote that of $|\psi\rangle$. In addition, recall that the entanglement entropy of a region in a tensor network state can be expressed in terms of the spectrum over the bonds connecting it to the rest of the system. Then the difference produced by applying $\mathcal{V}^\dagger$ is given by

$$\Delta\gamma = \sum_{\mathcal{V}_{AB}} S(b') + \sum_{\mathcal{V}_{BC}} S(b') - \sum_{\mathcal{V}_B} S(b') - \sum_{\mathcal{V}_{ABC}} S(b'). \tag{23}$$

We observe that this quantity is always nonnegative from a simple application of the principle of inclusion and exclusion, and moreover it is zero under the assumption that the spatial separation between A and C is much larger than Δ_0. With Eq. 21, we thus reproduce the desired result for states which can be described via MERA.

We remark that many states of interest which exhibit topological order are known to admit an exact representation, including the toric code and the quantum double model [13]. Moreover, it is known in 2D, that any translation invariant stabilizer code with local generators and code distance linear in the system size can be decomposed into a finite number of copies of the toric code [20], extending this result to a large class of states.

V. DISCUSSION

In this paper we have outlined methods for analyzing a broad class of quantum systems, which are obtained by extending renormalization group techniques from classical lattice models. In addition, we have shown the power of these models in studying the behavior of non-trivial topological phases of matter. We emphasize that our argument for the topological entanglement entropy holds only for states which can be exactly represented in MERA form. Furthermore, in practice, the topological entanglement entropy is difficult to compute for large regions as it requires obtaining the full reduced spectrum, and developing robust methods to characterize quantum phases remains an active area of research.

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