Quantum phase transitions in coupled atom-photon systems

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Quantum many-body phase transitions have attracted a great deal of theoretical and experimental study in both naturally-occurring systems like those in condensed matter and man-made systems created through cold atomic gases. Here, we discuss the possibility of observing quantum many-body phase transitions in lattices supporting localized photons. We review two proposals for observing quantum phase transitions in interacting photon systems: the Jaynes-Cummings lattice model and the closely related Rabi lattice model. Critical to the feasibility of these proposals is the effective interaction between photons mediated by strong coupling to atoms. The phase diagram of these models features a transition analogous to the Mott insulator-superfluid transition of Bose-Hubbard models. We expose these features through mean-field theory. We then extend the body of proposals by introducing two new lattice models: the Hopfield lattice model and the two-photon Rabi model, and exploring them with exact and numerical diagonalizations. Through these complementary methods, we find a few possibilities which were not noticed in the existing body of proposals: anisotropic superfluid phases, first-order transitions, and generally speaking, physics beyond the quartic Landau-Ginzburg Hamiltonian.

Phase transitions, whether classical or quantum, are generally a many-body phenomenon, and in order to access the full range of interesting physics of phase transitions, one needs interactions between the microscopic constituents. Since photons in an arbitrary linear medium are non-interacting, it is commonly believed that photons cannot be used to achieve phase transitions. Moreover, at the level of single quanta, photons in nonlinear media typically induce negligible nonlinearity, rendering nonlinear media insufficient for accessing phase transitions. Thus, in order to observe signatures of phase transitions in photonic systems, one would need either strong nonlinearity of a medium at the level of single quanta, or an effective interaction between photons mediated by matter.

It has long been known both theoretically and experimentally [1, 2] that atoms can be strongly coupled to photons in electromagnetic resonators. The resulting eigenstates of the coupled atom-photon system, polaritons, can be equal superpositions of matter and photon. This hybridization of light and matter can lead to an effective photon-number dependent repulsive interaction between photons that is mediated by the matter (called photon blockade) [3]. If we now have a lattice of these atom-cavity systems in which the photon can hop between sites via tunneling, it's clear that we can implement a lattice model that has hopping and a photon-number-dependent on-site interaction.

It was pointed out by Greentree *et al.* [4] that the aforementioned system is reminiscent of the Bose-Hubbard (BH) model in condensed matter physics, where one has a lattice in which bosons can hop from site to site while experiencing a boson-number-dependent repulsive on-site interaction. It is well-known in the BH model that there exists a transition between a Mott-insulating state (in which each lattice site has the same integer number of bosons) and a superfluid state, where the excitations are delocalized over the lattice and the average number of bosons on each site is non-integral [5]. Motivated by this analogy, they proposed the so-called Jaynes-Cummings (JC) lattice model, which is a lattice model in which the individual sites feature atoms strongly coupled to electromagnetic resonators and photons can hop from site to site. In this work, Greentree *et al.* performed a mean-field analysis of the system not unlike that performed in the seminal work on Mott insulator to superfluid transitions in BH models [5, 6]. It was found in the case of the JCL model that there exist two classes of distinct phases: Mott insulating phases, in which the order parameter is zero and an integral number of polaritons occupy each lattice site, and a superfluid phase with a uniform non-zero order parameter. In the ten years since this first proposal was put forth, there have been not only many elaborations on this particular proposal but several other lattice model proposals for realizing photon phase transitions, such as those in [7–12]. Nevertheless, this field is rather new, with many unexplored directions.

The organization of this work is as follows: first, we review the quantum theory of phase transitions in the mean-field approximation. With that developed, we review the two most popular proposals for realizing photonic phase transitions: the Jaynes-Cummings (JC) lattice model and the closely related Rabi lattice model. We show that the phenomenology of the JC lattice model is similar to the BH model, featuring a continuous phase transition from a Mott-insulating state of polaritons to a superfluid of polaritons. We then examine the Rabi model, a less approximate version of the JC model, and discuss the "relevance" of the new terms to the phase diagram of the system. After doing this, we introduce two new lattice models, not only to complement the existing body of literature on this topic, but to address a few loose ends left by existing works. We find qualitatively different mean-field physics than that realized in the previous works. We find a photonic lattice amenable to an exact solution at the mean-field level, potential for anisotropic superfluid phases, and first-order phase transitions; none of which exist in the previous proposals for photonic Mott insulators.

MEAN-FIELD THEORY OF QUANTUM PHASE TRANSITIONS

We start by introducing the class of models under consideration: we consider a two-dimensional lattice with N sites and $z_c N$ bonds, where z_c is the coordination network of each site. The identical lattice sites feature a single electromagnetic resonator constructed with linear dielectric media. We assume that each resonator supports only one mode. Each resonator is coupled to an atom with no restrictions on the strength of the coupling. The sites are coupled through the photon, which can tunnel from site to site via the evanescent field of the resonator mode. This system, schematically illustrated in Figure 1(a), corresponds to the Hamiltonian,

$$H = \sum_{\text{sites},i} H_i + \sum_{\langle i,j \rangle} \kappa a_i^\dagger a_j, \tag{1}$$

where H_i captures the on-site interaction between atoms and photons. In keeping with the bulk of the work on photonic phase transitions done thus far, we will analyze quantum phase transitions in the mean-field approximation; thus focusing only on phase transitions in order-parameter quantum field theories [13]. In this case, the criterion for a phase transition will be the same as in a classical field theory: a change of the order parameter from zero. The starting point in quantum mean-field theory is the expression of an arbitrary operator A as $\langle A \rangle + \delta A$, where $\langle A \rangle$ is the mean value of the operator and δA is an operator representing fluctuations about this mean. The actual approximation results from approximating $A \approx \langle A \rangle$. One consequence of this is that a product of operators AB can be expressed as:

$$\begin{aligned} AB &= \langle A \rangle B + (A - \langle A \rangle)(\langle B \rangle + \delta B) \\ &\approx \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle. \end{aligned} \tag{2}$$

Making this mean-field approximation, and defining the order parameter $\psi = z_c \kappa \langle a \rangle$ we arrive at an mean-field approximation of Equation (1): $H_{mf} = \sum_i H_{mf,i}(\psi)$, where

$$H_{mf,i}(\psi) = H_i - (a\psi^* + a^\dagger \psi) + \frac{1}{z_c \kappa} |\psi|^2,$$

is the on-site mean-field Hamiltonian. From this point on we suppress the site index in writing the on-site mean-field Hamiltonian. We note that this mean-field Hamiltonian has also been derived more rigorously in some works by the use of a Hubbard-Stratonovich transformation on the path integral of this system followed by a saddle-point approximation [5, 14]. The phases of the system depend non-trivially on the choice of the on-site Hamiltonian. To proceed, we will need to choose a specific Hamiltonian to analyze. In this work, we start with the JC and Rabi Hamiltonians discussed in [4, 14, 15], and then introduce two more Hamiltonians at the end of the work.

Mean-field phase diagram of the Jaynes-Cummings and Rabi lattice models

The first on-site Hamiltonian under consideration (H_i of Equation (1)) is the (resonant) Rabi Hamiltonian,

$$H_R = (\omega - \mu) a^\dagger a + (\omega - \mu) \sigma^+ \sigma^- + g(a + a^\dagger)(\sigma^+ + \sigma^-), \tag{3}$$

where $\sigma^\pm = \frac{\sigma_x \pm i \sigma_y}{2}$ are the usual Pauli raising/lowering operators. This effective Hamiltonian describes a dipole coupling (mediated by one-photon resonant interactions) between a two-level atom and a single photon mode. The two-level atom eigenstates are the ground state $|g\rangle$ of energy 0 and the excited state $|e\rangle$ of energy $\omega - \mu$. The photon mode is of frequency ω and the photonic eigenstates are just the Fock (oscillator) states $|n\rangle$ of energy $n(\omega - \mu)$ (neglecting zero-point energy). Note that we've introduced a chemical potential for the atom and photon. It will be necessary in order to get higher-order Mott insulating phases. Proposals for inducing a chemical potential for photons are discussed in [12].

An extremely common further approximation on this Hamiltonian is to discard the so-called 'counter-rotating' terms $g\sigma^+ a^\dagger$ and $g\sigma^- a$, leaving the Jaynes-Cummings (JC) Hamiltonian: $H_{JC} = (\omega - \mu) a^\dagger a + (\omega - \mu)\sigma^+\sigma^- + g(a\sigma^+ + a^\dagger \sigma^-)$ [23]. The single-site JC Hamiltonian is well-known to describe the time-dynamics of excited two-level atoms coupled to a photon mode when $\lambda \equiv \frac{2g}{\omega} \ll 1$.

In the original works on photonic Mott insulator-to-superfluid (MI-SF) transitions, this JC model was implemented on a lattice (through Equation (1)) to give the so-called JC lattice model. In [4], the phase boundaries in the $\kappa, (\omega - \mu)$ plane were derived through the following procedure: For each κ and $\omega - \mu$, numerically find the ground state energy of the mean-field Hamiltonian as a function of ψ, minimize it with respect to ψ, and identify the phase boundary with the line in parameter space where the minimizing psi is first different from zero. This procedure is very general and can easily be applied numerically to many systems, as was done in Greentree's work. That said, to have a more clear grasp of the physics of the problem, it is desirable to have an analytic method by which the phase boundary can be derived. In [14], such an analytic method was put forth, which involves approximating the mean field ground state energy to second order in time-independent perturbation theory, treating the single-site JC model as the unperturbed Hamiltonian and $-(a\psi^* + a^\dagger \psi) + \frac{1}{z_c \kappa} |\psi|^2$ as a perturbation. We review the calculation here.

In order to perform the calculation, we need the spectrum of the JC Hamiltonian. It can be seen that the JC Hamiltonian takes a block diagonal form constructed of 2×2 blocks which couple the states $|g, n\rangle, |e, n-1\rangle$ for $n = 1, 2, 3, ...$ The case $|g, 0\rangle$ corresponds to the ground state of the system in the absence of a chemical potential. This simplification reduces the problem to diagonalizing the following 2×2 matrix:

$$\begin{pmatrix} n(\omega - \mu) & \sqrt{n} g \\ \sqrt{n} g & n(\omega - \mu) \end{pmatrix}$$

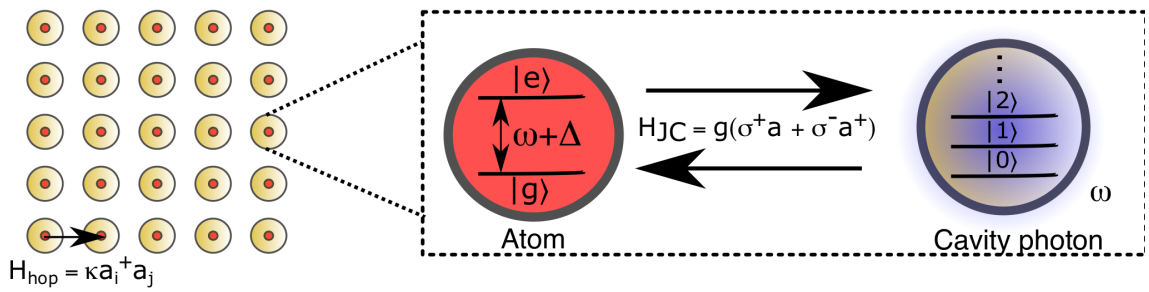

FIG. 1: **Jaynes-Cummings lattice model and single-site JC model.** (a) Schematic illustration of the JC lattice model: a two-dimensional lattice of electromagnetic cavity resonators with an atom embedded inside. A zoom-in is shown of the underlying single site constituents: a single two-level system (with transition energy $\omega + \Delta$) strongly coupled to the electromagnetic cavity mode of energy ω. The detuning Δ is assumed zero throughout this work.

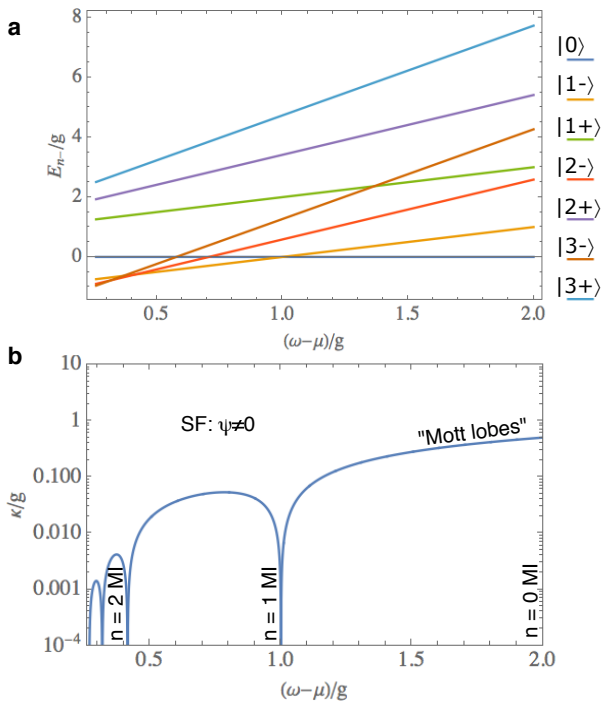

FIG. 2: **Mean-field phase diagram of one-photon JCL model.** (a) Level diagram of the single-site one-photon JC model as a function of the chemical potential. A crossover between different states being ground states corresponds exactly with the edges of adjacent Mott lobes. (b) Mean-field phase diagram of the one-photon JCL calculated through second-order non-degenerate perturbation theory on the mean-field Hamiltonian, plotted as a function of chemical potential and coupling. Underneath the Mott lobes correspond to ground states where an integer number of polaritons occupies each site. Above the lobes corresponds to the superfluid phase with non-zero order parameter $\psi \equiv z\kappa\langle a\rangle$.

for each n. The eigenstates are $|n\pm\rangle = \frac{|g,n\rangle \mp |e,n-1\rangle}{\sqrt{2}}$. The corresponding energies are $E_{n,\pm} = n(\omega-\mu) \pm g\sqrt{n}$. These states are commonly called polariton states because a polariton is a half-matter half-photon quasiparticle. The two states $|n\pm\rangle$ correspond to states with n polaritons. From the eigenvalues, it is clear that for any value of μ, one of the $|n-\rangle$ states is the ground state, which we illustrate in Figure 2(a), where we plot the energies E_{n-} as a function of $\omega - \mu$. For each value of $\omega - \mu$, we apply second order perturbation theory to the appropriate ground state. For values of $\omega - \mu$ where E_{n-} is the ground state, we have that the energy to second order in ψ is

$$E_0(\psi) = E_{n-} + |\psi|^2\left(\frac{1}{z_c\kappa} + R_n\right) + O(|\psi|^4), \quad (4)$$

where

$$R_n = \frac{1}{4}\sum_{\alpha=\pm1}\frac{(\sqrt{n}+\alpha\sqrt{n-1})^2}{(\omega-\mu)-g(\sqrt{n}-\alpha\sqrt{n-1})} + \frac{(\sqrt{n+1}+\alpha\sqrt{n})^2}{-(\omega-\mu)-g(\sqrt{n}-\alpha\sqrt{n+1})}$$

if $n \neq 0$ and $R_0 = \frac{1}{2}\sum_{\alpha=\pm1}\frac{1}{\mu-\omega+\alpha g}$. This leads to the condition for the phase boundary (to be justified later) for each n, $R_n = -\frac{1}{z_c\kappa}$. In Figure 2(b), we numerically solve these equations to get the phase boundary in the $\kappa, (\omega-\mu)$ plane. Our results are consistent with those obtained in [4, 14], provided $z_c = 3$, consistent with the assumption in those papers of a hexagonal JC lattice model. The phase boundary features so-called "Mott lobes" for each n. Underneath the Mott lobe denoted n, the order parameter is zero and the mean field ground state corresponds to a tensor product state in which each site has n polaritons. This is analogous to the BH model in which there are Mott lobes and the zero-order parameter phase corresponds to a fixed number of bosons per site. When we go above the Mott lobes, $\psi \neq 0$ and photons (and by virtue of coupling to the atom, polaritons) are delocalized through the lattice with non-integer mean occupation. The sharp vertical lines in the phase boundary separating different order Mott lobes correspond precisely to those values of $\omega - \mu$ for which the ground state changes (in the absence of hopping), as can be seen by comparing those points with the respective points on Figure 2(a). In particular, the many body ground state moves from $|(n-1)-\rangle^{\otimes N}$ to $|n-\rangle^{\otimes N}$ as we move from the $(n-1)$

MI to the n MI phase boundaries, justifying our characterization of the MI state as one with a fixed number of polaritons on each site. It is analogous to a disordered phase because $\psi = 0$ suggests that the photon hopping has no effect on the ground state.

We conclude our discussion of the JC lattice model phase boundary by noting that the above method is based on a similar method used in [6] in the context of deriving the MI-SF phase boundary for the original Bose-Hubbard model of condensed matter physics, where the mean-field ground state energy is found to *fourth* order in time-independent perturbation theory. In principle, for the calculation proposed in [14], time-independent perturbation theory would also need to be applied to fourth order. To understand why, we review the basic symmetries of the JC lattice model and use that to write down the general form of the ground state energy. The mean-field Hamiltonian is invariant under: $a \to ae^{i\phi}, a^\dagger \to a^\dagger e^{-i\phi}, \sigma^\pm \to \sigma^\pm e^{\mp i\phi}, \psi \to \psi e^{i\phi}$. In other words, there is a continuous $U(1)$ symmetry and $E_0(\psi) = E_0(|\psi|)$. Moreover, E_0 cannot depend on any odd power of $|\psi|$ as a result of the structure of the perturbation theory matrix elements. As a result, the ground state energy per site must take the form

$$E_0(\psi) = \frac{t}{2}|\psi|^2 + u|\psi|^4 + v|\psi|^6 + ...$$

Thus, second order perturbation theory extracts t, while fourth-order perturbation theory extracts u. It is thus now clear that deriving the phase boundary from setting the second order perturbation coefficient to zero implicitly assumes $u > 0$. In papers such as [4, 14, 15], this was not explicitly checked. However, when we plotted the energy density in the vicinity of a few points of the phase boundary numerically, the transition appeared to be continuous, validating the results obtained in the aforementioned works on the JC lattice model.

We now briefly review the extension of these results to the full Rabi model (JC with inclusion of counter-rotating terms) and the "relevance" of the counter-rotating terms on the shape of the phase boundary. This problem was first considered by Schiro *et al.* in [15]. As they note, the presence of the counter-rotating terms breaks the $U(1)$ symmetry and also the conservation of polariton number. To study the effect of the counter-rotating terms on the phase diagram in a controlled manner, they introduce a generalized Rabi model $H_{R,g} = \omega a^\dagger a + \omega\sigma^+\sigma^- + g(a\sigma^+ + a^\dagger\sigma^-) + g'(a\sigma^- + a^\dagger\sigma^+)$ with g' different from g (note the absence of a chemical potential). The true Rabi model has $g' = g$. To find the phase boundary, they apply the procedure of finding the ground state energy to second order in perturbation theory and setting the second-order coefficient equal to zero. As g' goes from 0 to g, the Mott lobes become suppressed and for $g' = g$, there is only a single phase boundary. We believe that the reason for this is that for $g' = 0$ (JC model), the eigenstates have a fixed number of polaritons because $N_i = \sigma_i^+\sigma_i^- + a_i^\dagger a_i$ is conserved. For even an infinitesimal g', this symmetry is broken and a well-defined polariton number cannot be associated to the many-body ground state (in the absence of hopping). What becomes clear from this discussion is that the counter-rotating terms are "relevant" in the sense of altering the phase diagram of the system. Despite that, the JC model provides useful insights, and can be of use in realizing BH models in photonics if one finds a Hamiltonian where the counterrotating terms disappear [24].

BEYOND THE QUARTIC ISOTROPIC LG HAMILTONIAN

Having reviewed the most popular proposals for implementation of phase transitions in photonic lattice models, we propose two new models and analyze the physics of phase transitions in these models at the mean-field level. To motivate our introduction of new models, we first note that there are some unsatisfying aspects of the works above on the JC and Rabi lattice models. In particular, it was assumed that the effective u derived from fourth-order perturbation theory is zero without having been checked. Nothing obvious forbids it from going negative since higher order dependences on ψ can induce the required stability. Of course, if this happens, then there is the potential for discontinuous phase transitions. It was also assumed in the work on the Rabi model that the order parameter can be chosen real, which does not appear to be true given that the counter-rotating terms spoil the $U(1)$ symmetry and reduce it to a simple parity symmetry. With this in mind, we ask the following questions:

1. Can anisotropy between the real and imaginary parts of the order parameter be realized (i.e., an anisotropic superfluid phase)? And more generally, what information can we get from mean-field theory without the use of perturbation theory?

2. Can discontinuous phase transitions be realized in these photonic lattice models?

In the last part of this work, we take two approaches to answering this question: developing exact solutions and numerical diagonalization methods. Both our fully analytic solutions and numerics suggest that the answer to the above two questions is affirmative. More generally, our results show that terms beyond the simple $\frac{t}{2}|\psi|^2 + u|\psi|^4$ LG Hamiltonian density play a role. We start our inquiry by introducing a model which we find can be diagonalized exactly once a mean-field approximation is made.

Exact Solutions and the 'Hopfield' Lattice Model

Inspired by the use of exact solutions to gain insight into field theories, we pose the question: are there any lattice models that can be exactly solved at the mean-field level? We note that in analogous problems in condensed matter physics like those of superfluidity and superconductivity, the corresponding lattice models can be exactly solved once a mean-field approximation is made [16]. In the JC lattice model or

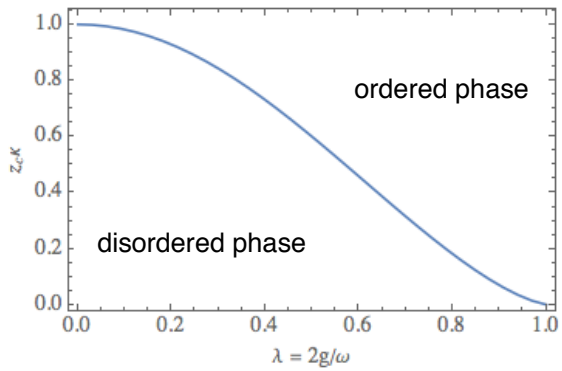

FIG. 3: **Mean-field phase boundary of the Hopfield lattice model.** Plot of the phase boundary of a lattice model of a harmonic oscillator coupled to a cavity mode derived by exactly diagonalizing the mean-field Hamiltonian. ω is taken to be 1 here.

Rabi model, this exact solvability does not occur due to the anharmonic spectrum of the atom (in particular, it is a two-level system). Noticing that harmonicity plays a role, we propose the following model: consider an array of photonic cavities (frequency ω) just like that of Figure 1. In each cavity, the photon interacts with matter in the form of an ideal one-dimensional harmonic oscillator (frequency ω, for simplicity) through the dipole moment $d = ex$ of the oscillator: $H = \omega a^\dagger a + \omega b^\dagger b - dE = \omega a^\dagger a + \omega b^\dagger b + g(a + a^\dagger)(b + b^\dagger)$. Here, the as are the operators of the photon and the bs those of the matter. As before, in the absence of photon hopping, this would be the Hamiltonian at each site. Including the hopping Hamiltonian and taking the mean-field approximation as before, we arrive at the mean-field Hamiltonian:

$$H_{mf}(\psi) = \omega a^\dagger a + \omega b^\dagger b + g(a + a^\dagger)(b + b^\dagger) - (\psi a^\dagger + \psi^* a) + \frac{1}{z_c\kappa}|\psi|^2. \tag{5}$$

Following the work on the Rabi model, we do not add a chemical potential. We call this model the *Hopfield* lattice model because of the close relation between coupled oscillators and the Hopfield model which models the quantum origin of dielectric properties. This Hamiltonian is exactly diagonalizable. We perform this in two steps. First, we diagonalize the terms not involving psi. Defining the photon and matter operators in terms of polariton operators following the transformation discussed in [17]:

$$\begin{aligned} a &= \frac{(r_+^{-1} + r_+)}{\sqrt{8}}A + \frac{(r_+^{-1} - r_+)}{\sqrt{8}}A^\dagger \\ &+ \frac{(r_-^{-1} + r_-)}{\sqrt{8}}B + \frac{(r_-^{-1} - r_-)}{\sqrt{8}}B^\dagger \\ b &= \frac{(r_+^{-1} + r_+)}{\sqrt{8}}A + \frac{(r_+^{-1} - r_+)}{\sqrt{8}}A^\dagger \\ &- \frac{(r_-^{-1} + r_-)}{\sqrt{8}}B - \frac{(r_-^{-1} - r_-)}{\sqrt{8}}B^\dagger \end{aligned} \tag{6}$$

with $r_\pm = \sqrt{1 \pm \frac{2g}{\omega}} \equiv \sqrt{1 \pm \lambda}$ we find that $\omega a^\dagger a + \omega b^\dagger b + g(a + a^\dagger)(b + b^\dagger) = \Omega_A(A^\dagger A + \frac{1}{2}) + \Omega_B(B^\dagger B + \frac{1}{2})$, where $\Omega_{A,B} = \sqrt{\omega^2 \pm 2g\omega} = \omega\sqrt{1 \pm \lambda}$. Expressing the mean-field Hamiltonian purely in terms of polariton operators, we get (neglecting zero-point energy)

$$\begin{aligned} H_{mf} &= \left(\Omega_A A^\dagger A - \alpha A - \alpha^* A^\dagger\right) \\ &+ \left(\Omega_B B^\dagger B - \beta B - \beta^* B^\dagger\right) + \frac{1}{z_c\kappa}|\psi|^2 \end{aligned} \tag{7}$$

with

$$\begin{aligned} \alpha &= \frac{(r_+^{-1} + r_+)\psi^* + (r_+^{-1} - r_+)\psi}{\sqrt{8}} \\ \beta &= \frac{(r_-^{-1} + r_-)\psi^* + (r_-^{-1} - r_-)\psi}{\sqrt{8}} \end{aligned} \tag{8}$$

Thus, the presence of the mean-field terms amounts to a simple shift of the oscillator equilibrium positions with no change in the polariton frequencies. Completing the square gives as the ground state energy (neglecting the ψ-independent zero-point energy):

$$\begin{aligned} E_0 &= -\frac{|\alpha|^2}{\Omega_A} - \frac{|\beta|^2}{\Omega_B} + \frac{1}{z_c\kappa}|\psi|^2 \\ &= \left(\frac{1}{z_c\kappa} - f_+(\lambda) - f_-(\lambda)\right)|\psi|^2 \\ &- (g_+(\lambda) + g_-(\lambda))\mathrm{Re}(\psi^2), \end{aligned} \tag{9}$$

with

$$f_\pm(\lambda) = \frac{1}{8\omega}\frac{\left(\sqrt{1 \pm \lambda} + \frac{1}{\sqrt{1\pm\lambda}}\right)^2 + \left(\sqrt{1 \pm \lambda} - \frac{1}{\sqrt{1\pm\lambda}}\right)^2}{\sqrt{1 \pm \lambda}}$$

and

$$g_\pm(\lambda) = \frac{1}{4\omega}\frac{\left(\sqrt{1 \pm \lambda} + \frac{1}{\sqrt{1\pm\lambda}}\right)\left(\frac{1}{\sqrt{1\pm\lambda}} - \sqrt{1 \pm \lambda}\right)}{\sqrt{1 \pm \lambda}}.$$

One can check that $g(\lambda) = g_+ + g_- > 0$ [25], and thus a real order parameter is preferred. This anisotropy is a clear

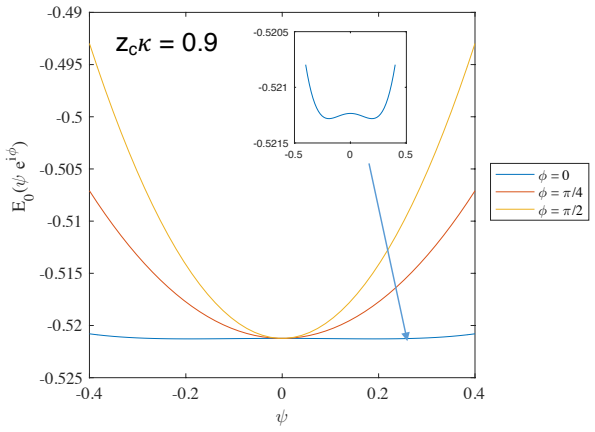

FIG. 4: **Anisotropic superfluidity for photons in the Rabi model.** Plots of the numerically obtained energy density of the ground state of the (one-photon) Rabi lattice model in the mean-field approximation. The energy density is plotted by scanning along rays in the complex plane with angles 0 (blue), $\pi/4$ (orange) and $\pi/2$ (yellow). It is clear here that the LG Hamiltonian associated with this energy density is anisotropic due to the $U(1)$ symmetry breaking by counter-rotating terms. Inset: zoom-in of the $\phi = 0$ energy density, which shows a minimum away from ψ=0, indicating a superfluid phase.

manifestation of the presence of counter-rotating terms in the coupled oscillator Hamiltonian. These are the same kind of counterrotating terms that appear in the Rabi model. This suggests that the same anisotropies are likely present in the Rabi model. We will corroborate this numerically in the next section.

Finally, we comment on the fact that this Hopfield lattice model maps to an (anisotropic) Gaussian model. Although strictly speaking, we can gain no information about the ordered phase due to the lack of quartic terms, we can at least find the point at which the model becomes unstable, indicating a phase transition from the disordered side [18]. Since $g > 0$, the condition for instability is

$$f_+ + f_- + g_+ + g_- = \frac{1}{z_c\kappa}.$$

Numerically solving this equation yields the 'phase boundary' of Figure 3. An encouraging aspect of this phase boundary is that there is only one phase boundary (i.e., no higher-order Mott lobes), just as in the case of the Rabi model, likely as a result of the lack of a chemical potential (in both models; in contrast to the JC model) and also the presence of counter-rotating terms.

One-photon Rabi model: anisotropy

As mentioned earlier, in the study of the Rabi model performed in [15], a real order parameter was assumed despite the fact that the $U(1)$ symmetry is broken. Having shown that $U(1)$ symmetry breaking led to anisotropy in the Hopfield model, it becomes plausible that the same anisotropy was present (although unexplored) in the Rabi lattice model. We now address this numerically. We implement a one-photon Rabi model numerically, diagonalize the mean-field Hamiltonian as a function of ψ and plot the resulting energy density as a function of ψ for different lines in the complex plane labeled by ϕ [26]. This result is summarized in Figure 4 for parameters choice of $\omega = 1, g = 0.2, z_c\kappa = 0.9$. While not a rigorous proof, our results suggest that a real order parameter globally minimizes the energy density. This is plausible given our fully analytical result for the Hopfield model, in which a real order parameter was also preferred. Nevertheless, the Hamiltonian was clearly anisotropic with respect to the phase, giving rise to an anisotropic superfluid phase. Moreover, the system no longer has goldstone modes, which is qualitatively quite different from the results for the JC model. Intuitively, our numerical result should suffice because the global minimum for ψ should only really occur on a high symmetry line such as $\phi = 0, \pm\pi/4, \pi/2$ given the way that ψ appears in the mean-field Hamiltonian. As becomes clear now, the development of an exact solution at the mean-field level helps to clarify the numerical results and make them more plausible.

Two-photon Rabi model: anisotropy and first-order transitions

Finally, we introduce another new model for which we find the mean-field solution numerically. Consider a situation in which an atom primarily interacts with a photon by resonant *two*-photon interactions (i.e., the primary real interaction between the atom and photon are two-photon absorption and emission processes). Recently, a number of schemes (such as that proposed in [19]) have been put forth for making the two-photon processes dominate the dynamics of the atom. This suggests that it could be justified to consider a (resonant) *two-photon Rabi Hamiltonian* [27]:

$$H_{R2} = \frac{\omega}{2}a^\dagger a + \omega\sigma^+\sigma^- + g(a^2 + a^{\dagger 2})(\sigma^+ + \sigma^-). \quad (10)$$

This Hamiltonian has been put forth by other authors (see [20–22]) but to our knowledge has never been considered in the context of a lattice model. We implement this Hamiltonian on the lattice, take the mean field approximation, and diagonalize the resulting Hamiltonian numerically. In Figure 5, we plot the energy per site as a function of ψ for different directions in the complex plane assuming $\omega = 1, g = 0.02$ for (a) $z_c\kappa = 0.43$, (b) $z_c\kappa = 0.453$, and (c) $z_c\kappa = 0.457$. As before, the lack of $U(1)$ symmetry creates an anisotropy in the complex plane. However, now either purely real or purely imaginary order parameters are preferred. We looked at the energy density for many different angles ϕ in the complex plane and found that $\phi = 0, \pi/2$ minimized the energy while $\phi = \pi/4$ maximized it. Interestingly, we find that somewhere in the vicinity of $z_c\kappa = 0.457$ there is a first-order phase transition to a non-zero order parameter. We were not able to devise an intuitive physical explanation for why the phase transition

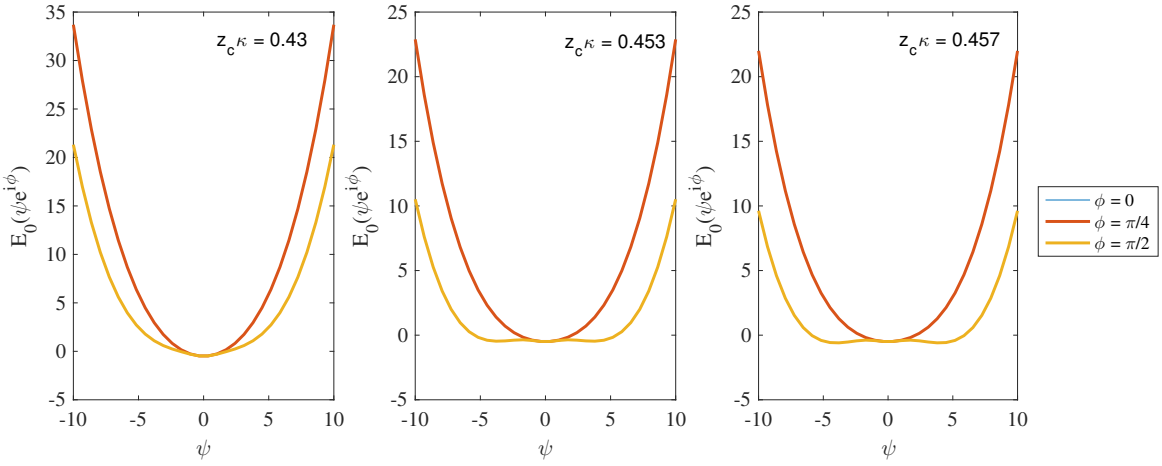

FIG. 5: **Two-photon Rabi model, anisotropic superfluidity, and first-order transitions.** Plots of the numerically obtained energy density of the ground state of the two-photon Rabi lattice model in the mean-field approximation. The energy density is plotted by scanning along rays in the complex plane with angles 0 (blue), $\pi/4$ (orange) and $\pi/2$ (yellow) for $g = 0.02\omega$ and (a) $z_c\kappa = 0.43$, (b) $z_c\kappa = 0.453$, and (c) $z_c\kappa = 0.457$. Note that in (b), there is roughly a double minimum $\psi = 0$ and $\psi \approx \pm 5$ and then abruptly moves to $\psi \approx \pm 5$; suggestive of a first-order transition. Further note that the blue and yellow lines lie exactly on top of each other. Unlike the one-photon case, we now see that either a real or imaginary order parameter is preferred.

becomes discontinuous here, but clearly the numerical results suggest an interesting direction of further inquiry. Ffrom a mathematical perspective it results from the atom-photon interaction being a cubic form rather than a quadratic form as in the previous cases considered. Another interesting thing to note about the energy densities plotted in Figure 5 is that when we fit it to a polynomial (of the Landau-Ginzburg form), terms up to *eighth* order were needed to get a sufficient fit, potentially invalidating the assumptions that back the perturbative procedures applied in the JC and Rabi lattice models reviewed here.

CONCLUSION

To summarize, we have reviewed the initial proposals for observing phase transitions in interacting photon systems, and attempted to supplement the physics of the existing models with two new models: a so-called 'Hopfield' lattice model, which was amenable to an exact solution once a mean-field approximation was made, and a two-photon Rabi lattice model. We found by looking at these models (and also the one-photon Rabi model in more detail) that a number of interesting phenomena beyond the simple isotropic, quartic LG Hamiltonian. We found the possibility for realizing: Gaussian models (through coupling the photon to an oscillator Hamiltonian), anisotropic superfluid phases (as a result of $U(1)$ symmetry breaking by counter-rotating terms), and discontinuous phase transitions when two-photon interactions are involved (whose origin we could not find a simple intuition for). Our reason for exploring several different models was to make apparent the fact that many different Hamiltonians can be realized when one looks at different situations in the physics of light-matter interactions. Other important directions of study in this field will be understanding the relevance of: higher-order resonant multi-photon interactions, multi-mode models, and also multipolar interactions beyond the electric dipole approximation taken in this work and other works.

This field of finding photonic analogs of condensed matter systems is rather new, and so we hope that the introduction of two new models can help to paint a broader picture of what might be possible. It's worth noting that the main limitation of the work performed here (and in prior literature) is the mean-field approximation. Although we know mean-field approximations can often give qualitatively reasonable results, it will be important to see the body of results thus far corroborated by Monte Carlo techniques, RG analysis, etc. One thing is clear; although experiments in realizing photonic phase transitions seem far away, it could be quite profitable to realize because it is much easier to controllably induce different Hamiltonians (and different lattice models) than in condensed matter analogs. The diversity of light-matter interactions in quantum electrodynamics should in principle allow for interactions to be implemented in lattice models that could not be implemented in condensed matter analogs. This will ensure that the field doesn't just become a 'repeat' of the study of quantum phase transitions in condensed matter physics.

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[23] This approximation, called the rotating-wave approximation (RWA) is generally valid for $\lambda \equiv \frac{2g}{\omega} \ll 1$ (the 2 is conventional), and its validity in this regime comes from the fact that the discarded terms represent non-resonant processes while the retained terms are resonant terms corresponding to 'real' energy-conserving emission and absorption processes.

[24] Or are simply less than g, because the lobes are gradually suppressed as g' is increased towards g.

[25] λ is assumed to be positive, as is conventional. Moreover, $\lambda < 1$, otherwise one of the frequencies becomes imaginary.

[26] An important note regarding numerical implementation: as is necessary in a numerical diagonalization, we have to truncate the photon Fock space. The truncation can lead to modification of the eigenvalues when the magnitude of the order parameter corresponds roughly to the maximum number of photons in the Fock space. We found that for the values of ψ considered in Figures 4 and 5, a Fock space dimension of 600 was sufficient to ensure convergence of the results. For much larger ψ, the eigenstates obtained will probe higher and higher photon number and the results would become unreliable.

[27] We are assuming that the parity of the excited and ground states is such that any one-photon process is strictly forbidden, leaving only two-photon interactions in the effective Hamiltonian.