Potts Model, Duality and Percolation

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We present formal description of the Potts model and the multi-site Potts model. Then using series expansions, we derive the relation between coupling strengths of Potts model on a lattice and its dual. We explain the percolation on a lattice and its connection to the Potts model and general Potts model.

I. INTRODUCTION

Potts model[1] and its connection to the percolation problem has been studied widely[2, 3]. There have been wide studies of duality of Potts model and relation of Potts model to so many other models. Good review of most of them can be found in [2]. The relation between Potts model and percolation has been explained well in [3, 4]. [3] has used the duality of percolation and generalized Potts model to derive the duality relation for the generalized Potts model itself.

We will review the low-temperature and high-temperature series expansion discussion for deriving the duality transformation of Potts model, and then describe general Potts model as introduced in [3]. Finally will explain the Potts-percolation duality on ordinary and general Potts model.

II. MODEL

Potts model is a special case of the *clock model* [5]. In a clock model, on each site of a lattice, there is a spin that can take q different values ($s_i = 1, \cdots, q$) (especially $q = 2$ for Ising model) and interactions between different spins are invariant under changing involved spins altogether in $\mod q$ (e.g. for interactions involving two spins $-\beta\mathcal{H}_I = \sum_{\langle i,j\rangle} J(|s_i - s_j| \mod q)$).

This model describes *Potts model* when there is a full permutation symmetry as well. In this case, the only energy cost can be related to different spins having the same value or not. In the case of the *ordinary Potts model* there is only two-spin interactions. In interactions with more than two spins involved, which happens in the *general Potts model*, in general there can be different energy costs for different number of spins having the same value.

We always assume that there is no external magnetic field.

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III. DUALITY OF ORDINARY POTTS MODEL

Consider a lattice $G = (V, E)$, where V and E are sets of vertices and edges respectively. Here we consider planar lattices with the coordination number, e.g. number of the neighbours of a site, λ. The dual lattice of G, denoted by D, is constructed by having the centers of faces of G as vertices, and bonds cutting edges of G as edges. The dual lattice for a square lattice, is another square lattice. But the dual lattice for a hexagonal lattice is a triangular lattice and vice versa, and the dual for a kagome lattice[6] is a dice lattice[7] and vice versa.

We put a $q-$valued spin on each vertex and each edge shows two-site interaction between the spins of the corresponding vertices. Time-dependent strength of the interaction, K, is assumed to be independent of the edge.

The partition function of an ordinary Potts model with $q-$valued spins s_i on a lattice with nearest neighbour interactions can be written as

$$Z_G(q, K) = \sum_{\{s_i\}=1}^{q} \exp\left(K \sum_{\langle i,j\rangle} \delta_{s_i,s_j}\right). \qquad (1)$$

($\langle i, j\rangle$ means that there exists an edge between spins on the vertices i and j.)

Here, we construct a low-temperature series expansion for the lattice G and a high-temperature series expansion for its dual, lattice D. In low temperatures ($K \gg 0$), we can expand the partition function starting from the zero temperature partition functions, corresponding to the state that all spins are aligned, and then writing excitations as *islands* of other spins among the sea of aligned spins. The expansion will be as follows[2]:

$$Z_G(q, K) = qe^{|E|K}\left(1 + \sum_{s=\lambda}^{|E|} a_s e^{-sK}\right),$$

here $a_s = \sum'_{n_i} a(\{n_i\}, s)$, that $a(\{n_i\}, s)$ is the number of spin configurations in which there is n_i sites in the i'th state and there are s bonds that connect sites that are in different spin states, and $'$ on sum is for the condition $\sum n_i = N$.

The overall factor corresponds to the zero temperature configuration, e.g. all spins aligned. There is the q prefactor because this alignment can be in any of spin states.

The first exitations are configuration that are achived with flipping just one spin, which will break λ bonds.

For high-temperature expansion on the lattice D, we need to write the partition function in a new fashion, such that for each bond there are two conditions. One can write 1 as [2]

$$Z_D(q,K^*)=\sum_{\{s_i\}=1}^{q}\prod_{\langle i,j\rangle}(1+v\delta_{s_i,s_j})), \tag{2}$$

where $v=e^{K^*}-1$. It seems useful for a high-temperature expansion, since $v\ll 1$ for $0<K\ll 1$. But for an ideal such expansion, we need that after writing down the products and summing over s_i's, only terms describing loops survive. But $\sum_{s_i}\delta_{s_i,s_j}\neq 0$ and vertices with degree one will survive. So we need to write 1 in another fashion as follows

$$Z_D(q,K^*)=\sum_{\{s_i\}=1}^{q}\prod_{\langle i,j\rangle}(t(1+f_{ij})),$$

in which $t=\frac{q+v}{q}$ and $f_{ij}=\frac{v}{v+q}(-1+q\delta_{s_i,s_j})$. Now, f_{ij} is suitable in the required way and $\sum_{s_i}f_{ij}=0$. Therefore we can use it to construct high-temperature expansion. It contains closed loops that divide the surface of faces of D into different regions. The coefficient for each configuration is proportional to $\frac{v}{v+q}$ to the power of the length of the loops that it contains. This closed loops can be seen as islands of sites in the original lattice, G. So the high-temperature series expansion for the lattice D, is up to a trivial coefficient, equal to the low-temperature series expansion of the lattice G, if we substitute $\frac{v}{v+q}=e^{-K}$. Hence,

$$e^K=\frac{v+q}{v}=1+\frac{q}{e^{K^*}-1},$$

thus,

$$(e^K-1)(e^{K^*}-1)=q. \tag{3}$$

IV. DUALITY OF GENERAL POTTS MODEL

Essam [3] defines a generalized Potts model that can be used for describing multisite interactions. For this purpose, he denotes sites of G that have spins on them with S, and adds to them the interaction vertices I. So $G=(V,I)$ is a bipartite lattice that $V=S\cup I$. and each vertex in S is connected only to vertices of I and each vertex of I is connected only to the vertices of S. We assume each vertex in I is connected to γ vertices in S, e.g. we have $\gamma-$site interactions and $\gamma=2$ is the ordinary Potts model.

For having the complete model, we need to know the energy costs of different configurations. Since we still consider the case with no external field, the simplest energy cost model is when there is time-dependent cost K_γ for not having all the spins connected to one vertex of I in the same state. Thus, the partition function will be

$$Z_G(q,K_\gamma)=\sum_{\{s_i\}=1}^{q}\exp\left(K_\gamma\sum_{i\in I}\delta_{s_{i,1},s_{i,2}}\delta_{s_{i,2},s_{i,3}}\cdots\delta_{s_{i,\gamma-1},s_{i,\gamma}}\right) \tag{4}$$

Now we want to construct the dual lattice of G. We denote it by $G^*=(V^*,E^*)$ and $V^*=S^*\cup I$. In other words, the vertices of G^* have the same interaction vertices of G, but spin sites now are the centers of the faces of G. For E^*, we connect each member of S^*, that is the center of a face, to the interaction vertices on the boundary of that face.

In [3] is shown that the duality relation between K_γ and K^*_γ for the general Potts model is

$$(e^K_\gamma-1)(e^{K^*_\gamma}-1)=q^{\gamma-1}, \tag{5}$$

which for the case of $\gamma=2$ is just 3.

V. RELATION TO THE PERCOLATION

In III we wrote the partition function in the fashion of 2. Even though it wasn't useful for high-temperature series expansions, it is still useful to use it to write the partition function in the form of products of that terms as follows

$$Z(q,K)=\sum_{G'}v^{n_b}q^{n_c}.$$

Here G''s are sub-lattices of G that each has different number of bonds included in itself. n_b is the number of bonds of G' and n_c is the number of clusters of it, such that different clusters are not connected at all.

In the case of $q=1$, this partition functions, up to a prefactor, describes different configurations of occupations of the lattice, where the odds of occupation, $\frac{p}{1-p}$, is v, therefore

$$p=\frac{v}{1+v}.$$

This is just the bond percolation model.

We can have a bond percolation model on lattice with multi-site interaction using the general Potts model. We need to define the probability of occupation of each vertex of S equal to 1 and then use the fashion of 2 to reshape the partition function of the general Potts model (4), and at the end defining the occupation probability of each vertex of I equal to $p=\frac{v}{1+v}$.

Therefore, we've seen that Potts model and general Potts model, described in III and IV, can be used to derive percolation phase transitions. Many numerical simulations can be done on the Potts model on different lattices [8, 9] and the results of them can be used in percolation problems.

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