Renormalization of Chiral Sine-Gordon Theory

Sungjoon Hong*
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
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Chiral Sine-Gordon Theory appears naturally in the double layer Quantum Hall setting. It describes simplest possible interaction Hamiltonian between chiral edge modes of each layer. Unlike well-known Sine-Gordon theory which shows Kosterlitz-Thouless type flow, chiral version of Sine-Gordon theory behaves differently under RG. This can explain that the relevance of tunneling between edge modes depends on bulk topological structure, and the during this process, we can see the power of changing between Boson and Fermion representation.

I. INTRODUCTION

Measuring tunneling between edge modes of double-layer quantum Hall system is a great way (and the only way) to experimentally investigate the bulk topological structure of quantum Hall system. Quantum Hall edge states are first explained by Halperin [1] in IQH(Integer Quantum Hall) system, and by Wen[2,3] in FQH(Fractional Quantum Hall) system. According to the Wen's hydrodynamical approach of fractional quantum Hall edge, since bulk is gapped, the only low lying excitations below bulk gap are surface waves from the deformation of FQH droplet. In this approach, edge modes are described by chiral Boson fields, which is in some sense surprising because electrons are fermions. 'Chiral' means that edge states propagate in one direction, for we broke time reversal symmetry by applying magnetic field.

Using this boson picture, we can derive that tunneling is described by Hamiltonian of the form

$$H_{\chi SG}[\phi] = \int dx[\frac{1}{4\pi}(\partial_x \phi)^2 - \kappa \cos(\beta\phi)] \tag{1}$$

where ϕ is chiral Boson field which satisfies $[\phi(x), \phi(x')] = i\pi sgn(x - x')$.

This at first looks similar to (ordinary) Sine-Gordon Theory, whose RG(renormalization group) flow is well-known to be Kosterlitz-Thouless type. Though we will not derive it here, several methods can be used to calculate this(e.g.[4,5]). There were once some belief that our Chiral Sine-Gordon(χSG) model can be mapped into ordinary Sine-Gordon(SG) theory, but we now know that this is wrong. The RG behavior of χSG theory is completely different(and somewhat much more simpler) than SG theory, and it shows that relevance of tunneling between double-layer edge modes changes according to bulk topological structure.

This line of approach is first done by Kane, Fisher, and Polchinski [6] in single layer quantum Hall system with filling fraction $\nu = \frac{2}{3}$, which has two inversely propagating edge modes. In that paper, they exactly solved the tunneling between two edge modes by giving additional hidden symmetry imposed by random impurity. After that, Naud, Pryadko, and Sondhi [7] gives nearly full analysis of χSG theory in cases which can be applied to quantum Hall bilayer problem.

II. DESCRIBING BILAYER FQH EDGE STATES

II.1. Bilayer QH wavefunction and Hydrodynamics Approach of edge modes

Here I will briefly state Halperin[1]'s result and derive Wen[2,3]'s result about FQH edge state. Halperin showed that double layer(Abelian) QH wavefunction is

$$\Psi(z_{i\alpha}) = \prod_{\alpha<\beta}(z_{1\alpha} - z_{1\beta})^m (z_{2\alpha} - z_{2\beta})^{m'} \prod_{\alpha,\beta}(z_{1\alpha} - z_{2\beta})^n \times e^{-\sum_{i,\alpha}|z_{i\alpha}|^2/4} \tag{2}$$

Here, filling fraction of each layer is $\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \frac{1}{mm'-n^2}\begin{pmatrix} m'-n \\ m-n \end{pmatrix}$.

To introduce edge state theory, consider first an FQH droplet with filling fraction ν. Since bulk is gapped, the low lying excitations are edge waves(deformation of edge), and can be described by one-dimensional density $\rho(x) = nh(x)$, $n = \nu/2\pi l_B^2$ as

$$\partial_t \rho + v\partial_x \rho = 0 \tag{3}$$

So the Hamiltonian is

$$H = \int dx \frac{1}{2} e\rho E h = \int dx \pi \frac{v}{\nu}\rho^2 \tag{4}$$

After quantizing this Hamiltonian, we get

$$[\rho_k, \rho_{k'}] = \frac{\nu}{2\pi} k\delta_{kk'}, \quad k, k' = integer \times \frac{2\pi}{L} \tag{5}$$

$$H = 2\pi \frac{v}{\nu}\sum_{k>0}\rho_{-k}\rho_k \tag{6}$$

This algebra is often called $U(1)$ Kac-Moody Algebra. If filling fraction is not of Laughlin type, we can use hierarchical construction to generalize. By letting $\rho = \frac{1}{2\pi}\partial_x u$,

* sungjoon@mit.edu

we can get

$$[u_i(t,x), u_j(t,x')] = i\pi K_{ij} sgn(x-x') \tag{7}$$

where u_i is chiral Boson field and $K_{ij} = \begin{pmatrix} m & n \\ n & m' \end{pmatrix}$. Also, using our knowledge of Bosonization, the vertex operator which creates electron is

$$\Psi_i^\dagger(x) \propto e^{-iu_i(x)} \tag{8}$$

II.2. Constructing Tunneling Hamiltonian

Let's consider only symmetric case, $m = m'$, and $v = v'$, since mapping to χSG theory exists only in symmetric case. First, as can be seen in [6], we can transform us into new fields called charge and neutral mode ϕ_c, ϕ_n.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{m+n} & -\sqrt{m-n} \\ \sqrt{m+n} & \sqrt{m-n} \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_n \end{pmatrix} \tag{9}$$

We can easily see that new charge and neutral field also satisfies similar commutation relation $[\phi_i(x), \phi_j(x)] = i\pi sgn(x-x')$, where i,j index runs through c,n. Now, using the electron operator described before, I will construct simplest single-electron tunneling Hamiltonian as follows(we used Fermion normal ordering).

$$H_1 = \lambda_0 \int dx [: \Psi_1(x)\Psi_2^\dagger(x) : +h.c.] \tag{10}$$

Applying $\Psi_i^\dagger(x) \propto e^{-iu_i(x)}$ and (9), we can get full Hamiltonian.

$$\begin{aligned} H = \int dx [&\frac{1}{4\pi} v_c : (\partial_x \phi_c)^2 : + \frac{1}{4\pi} v_n : (\partial_x \phi_n)^2 : \\ &+ \frac{2\lambda}{(2\pi a)^{\beta^2/2}} \cos(\beta\phi_n)] \end{aligned} \tag{11}$$

where $\beta = \sqrt{2(m-n)}$, and $\lambda = \lambda_0 L^{-n}$ has length dimension $(\beta^2/2 - 2)$, and a is length cutoff. We can see that full Hamiltonian is the sum of free chiral boson theory of charged mode and Chiral Sine-Gordon theory of neutral mode.

III. RENORMALIZATION GROUP FLOW OF χSG THEORY

Write our Hamiltonian's nontrivial part(except trivial quadratic charge mode) in Euclidean action form, with dimensionless coupling $\hat{\lambda} = 2\lambda(2\pi a)^{2-\beta^2/2}$

$$\begin{aligned} S_{\chi SG}[\phi_n] = \int dx d\tau (&\frac{1}{4\pi}[i(\partial_\tau \phi_n)(\partial_x \phi_n) + (\partial_x \phi_n)^2] \\ &+ \frac{\hat{\lambda}}{(2\pi a)^2} \cos(\beta\phi_n)) \end{aligned} \tag{12}$$

compared with standard SG theory.

$$\begin{aligned} S_{SG}[\phi_n] = \int dx d\tau (&\frac{1}{4\pi}[(\partial_\tau \phi_n)^2 + (\partial_x \phi_n)^2] \\ &+ \frac{\hat{\lambda}}{(2\pi a)^2} \cos(\beta\phi_n)) \end{aligned} \tag{13}$$

We can see that after writing down the equation of motion, SG theory has two modes propagating opposite direction $i\partial_\tau \phi_n + \partial_x \phi_n = 0, i\partial_\tau \phi_n - \partial_x \phi_n = 0$, where χSG theory has only one, $i\partial_\tau \phi_n + \partial_x \phi_n = 0$. This coincides with our knowledge that edge modes of quantum Hall system propagates one way.

Momentum shell RG process to SG theory is done in several literature, like Wiegmann[5]. Applying same method to χSG theory, we get, in first order of $\hat{\lambda}$, the following.

$$\frac{d\hat{\lambda}}{dl} = (2 - \frac{\beta^2}{2})\hat{\lambda}, \quad \frac{d\beta^2}{dl} = 0 \tag{14}$$

compared with original SG theory RG flow.

$$\frac{d\hat{\lambda}}{dl} = (2 - \frac{\beta^2}{2})\hat{\lambda}, \quad \frac{d\beta^2}{dl} = -c\hat{\lambda}^2 \tag{15}$$

We can see the 'in lowest order'(tree level), tunneling between bilayer quantum Hall system is relevant if $\beta^2 = 2(m-n) < 4$, and irrelevant if $\beta^2 = 2(m-n) > 4$, and marginal if $\beta^2 = 2(m-n) = 4$. The phase transition point between relevant tunneling and irrelevant tunneling is a line $\beta^2 = 4$. Remarkable thing in χSG theory is that β doesn't flow, unlike ordinary SG theory.

IV. WHY TREE LEVEL RESULT IS EXACT? FERMION REPRESENTATION

So natural question arising is 'is this RG flow exact in all order?', especially in the marginal case $\beta^2 = 4$. For this purpose, we will use Bosonization and re-Fermionization technique which is well explained in [8], and below we will follow [7] and [8] to get Fermionic representation of χSG theory. First, let's define a chiral Dirac Fermion field

$$\psi(x) = \sqrt{\frac{1}{2\pi a}} e^{i\phi_n(x)} \tag{16}$$

Now,

$$\psi(x)\psi(x+e) = e\psi(x)\partial_x\psi(x) + o(e^2) \tag{17}$$

where we used $\psi(x)\psi(x) = 0$ since ψ is Fermionic field. Also,

$$\begin{aligned} \frac{1}{L} : e^{i\phi_n(x)} :: e^{i\phi_n(x+e)} := &\frac{2\pi a}{L^2} : e^{2i\phi_n(x)} : + \frac{2\pi i e}{L^2}(: e^{2i\phi_n(x)} : \\ &- a : [\partial_x \phi_n(x) + \frac{3\pi}{L}] e^{2i\phi_n(x)} :) \\ &+ o(e^2, a^2) \end{aligned} \tag{18}$$

Since RHS of (18) is normal ordered, we can safely take limit $a \to 0$. Compare (17) and (18), we get what we need.

$$\psi(x)\partial_x\psi(x) = \frac{i}{2\pi a^2}e^{2i\phi_n(x)} \tag{19}$$

So using (19), Fermionic representation of our χSG Hamiltonian with $\beta^2 = 4$ is

$$H_{\chi SG} = \int dx : [-iv_n\psi^\dagger\partial_x\psi - i\frac{\lambda}{2\pi}(\psi^\dagger\partial_x\psi^\dagger + \psi\partial_x\psi)] : \tag{20}$$

Surprisingly enough, this Hamiltonian is quadratic in chiral Dirac field Ψ. To see more physical picture, we further transform this Dirac fermion into two Majorana component, i.e.$\psi(x) = (\chi_1(x) + i\chi_2(x))/\sqrt{2}$, then

$$H_{\chi SG} = -\frac{1}{2}\int dx : [i(v_n+\frac{\lambda}{\pi})\chi_1\partial_x\chi_1 + i(v_n-\frac{\lambda}{\pi})\chi_2\partial_x\chi_2)] : \tag{21}$$

So two Majorana fields are totally decoupled and each has different speed modified by tunneling parameter λ. We can see from this Fermionic representation that since Hamiltonian is quadratic, after doing RG, high frequency modes and low frequency modes are decoupled, which means tree-level RG is EXACT. Therefore, in our fixed line $\beta^2 = 4$, we can conclude that tunneling is exactly marginal.

V. CONCLUSION

We reviewed theories to construct χSG Hamiltonian which describes interlayer tunneling of bilayer quantum Hall system. From simple momentum-shell RG process, we can conclude that relevance of tunneling depends on bulk topological structure of both layer (described by number m, n). The RG flow of χSG theory is different from ordinary SG theory, and β^2 does not flow under RG. Also, Bosonization and re-Fermionization technique show that in marginal β, our tree level(first order) result is exact in every order, so $\beta^2 = 4$ is exactly marginal. This is significant in that $\beta^2 = 4$ can be experimentally tested, since $2(m-n) = 4$ includes $m = m' = 3, n = 1$ (using Halperin's notation,[331]state) state, which is experimentally made.

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