Vanishing gaps in random transverse Ising model: Difficulty and new strategy for the Quantum Adiabatic Algorithm*

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The conventional Quantum Adiabatic Algorithm(QAA) encounters problem of exponential closing gaps when applied to a restrictred class of MAX 2-SAT problem. This exponential closing gap is exactly caused by the phase transition in random transverse Ising spin model predicted by Fisher *et.al.* [1, 2]. We propose some strategy that will hopefully avoid this exponential closing gap, promising a polynomial running time of QAA for a restrictred class of MAX 2-SAT problem.

I. QUANTUM ADIABATIC ALGORITHM

By making use of the adiabatic theorem, quantum adiabatic algorithm[3](QAA) allows us to start from the ground state of simple Hamiltonian $H(0)$ and arrive at the ground state of a complicated Hamiltonian H_p. If we properly design the final Hamiltonian, we can in principle solve many NP complete problem[4]. A classical example that is being theoretically and experimentally studied is the MAX 2-SAT problem[5].

The MAX 2-SAT problem with N variables $x_1, \cdots, x_N \in \{0, 1\}$ and M clauses(each clause is a logical OR of two literals, where a literal is a variable or its negation) asks the following question: *Given conjunctive normal form(CNF)*

$$F(\vec{x}) \equiv (x_1 \vee x_2) \wedge (x_1 \vee \neg x_3) \wedge \cdots \wedge (x_N \vee x_6) \quad (1)$$

find the variables' assignment that maximize the number of clauses being satisfied. The hardness of the problem is related to the clause density $\alpha \equiv M/N$[6], a CNF with smaller α is relatively easy.

To solve the problem by QAA, as routine(which we argue in Section. VI, may not be a good choice), the initial Hamiltonian is chosen as

$$H(0) = \sum_{j=1}^{N} k_j S_j^x \quad (2)$$

and for MAX 2-SAT problem the final Hamiltonian is

$$H_p = \sum_{j=1}^{N} h_j S_j^z + \sum_{i,j=1}^{N} J_{ij} S_i^z S_j^z \quad (3)$$

which is an infinite range SK spin glass model with a local field(Eqn (4.4) in Ref.[7]). The field h_j and coupling J_{ij} are determined by the details of the MAX 2-SAT problem(see Appendix. A). Note that the initial Hamiltonian can be chosen differently, as long as its ground state is known and easy to prepare. In fact how to choose an initial Hamiltonian to optimize the batic algorithm is not trivial at all.

While the adiabatic path in the Hamiltonian space connecting the initial Hamiltonian $H(0)$ and the final Hamiltonian H_p can be chosen specificly to optimize the algorithm's performance[8, 9], here for simplicity we only consider the following path

$$H(t) = (1 - \frac{t}{T})H(0) + \frac{t}{T}H_P \quad (4)$$

The QAA's running time is constrained by Landau-Zener transition, an upper bound can be given as

$$\mathcal{T} \ll \hbar |\max_t V_{10}(t)|/(\Delta E_{\min})^2 \quad (5)$$

where $\Delta E_{\min}$ is the minimum of the first excitation gap of Eqn. (4) in time $[0, T]$; $V_{10}(t) = \langle\psi_1|dH/dt|\psi_0\rangle$ is the matrix element between the first excited state and ground state. Usually the gap square $\Delta E_{\min}^2$ is the key part determing the running time; if there is no gap closing at any system size along the path of $H(t)$, then a constant running time is expected; if there is a closing gap powerlaw small of the system size, then a powerlaw running time is expected; if there is a closing gap exponentially small of the system size, then exponential running time is expeted.

II. QUANTUM ISING SPIN UNDER TRANSVERSE FIELD

In order to understand QAA's running time, we need to know the spectrum of the Hamiltonian given in Eqn. (4). Since it is not easy to solve, we need to simplify the Hamiltonian. By placing some restriction on MAX 2-SAT problem(for details see Appendix. A), we require that in Eqn. (3) interaction is restricted to nearest neighbour and the field in z direction is zero(this field destroys the solvability[2]).

To be consistent with conventional formulism, we make the time dependence implicit in the external field, i.e. $\Gamma_i \propto \frac{T-t}{t}$. We arrive at the quantum Ising spin with

*This is a term paper for Class 8.334 at MIT, some of the work is done not knowing Fisher et.al.'s results[1, 2], in the end similar results are found by the author, but the paper is orgnized in a form of citing their results first

random transverse field Γ_i on spin i and coupling J_i between neighbouring spin i and $i+1$, $i = 1, \cdots, N$, where N is the number of spins

$$H(t) = -\sum_{i=1}^{N}(\Gamma_i S_i^x + J_i S_i^z S_{i+1}^z) \tag{6}$$

Due to gauge invariance(see Appendix. B), for nearest neighbour coupling chain we can always choose all transverse field and coupling to be positive, i.e. $\Gamma_i > 0, J_i > 0$. Without further knowledge of what specific value of the field Γ_i and coupling J_i realistic MAX 2-SAT will give, in order to consider the typical case, we choose Γ_i's and J_i's to be random indepedent numbers drawn from certain distribution(iid). As the detailed form of distribution isn't important[10], we choose Γ_i uniform in $[\Gamma_{\min}, \Gamma_{\max}]$ and J_i uniform in $[J_{\min}, J_{\max}]$.

With the restrictions on, one can show(see appendix.A) that the MAX 2-SAT is balanced(with equal number of x_j and its negation $\neg x_j$ in the CNF, $\forall j$) and the clause density $\alpha > 1/2$. This restricted MAX 2-SAT is likely to be among the easy cases, in fact one can see from Eqn. (6), $H(T)$ is a trivial nearest neighbour Ising chain with a ground state of all spins alligned.

III. SOLVING THE EIGENVALUES

Before solving the eigenspectrum of Eqn. (6), it is worthy to consider the symmetry property of the Hamiltonian. Consider spin flip operator $F \equiv S_1^x S_2^x \cdots S_N^x$, a little algebra shows that [F,H]=0. Thus there is a spin-flip degeneracy in the eigenvalues of the system, and the eigenstates are divided into two sectors by F. However, since the evolution of the Hamiltonian dH/dt generally will not be proportional to F, thus the matrix element between the degenerate states are zero. Consequently when considering the running time in Eqn. (5), we only need to consider one sector of the non-degenerate states.

Using Wigner-Jordan transform, the system can be mapped to a non-interacting fermion system[11, 12]. Use the c-cyclic boundary condition, the eigenvalue problem can be reduced to solving the $N \times N$ positive semi-definite matrix $A^T A$'s eigenvalues $\lambda_k, k = 1, \cdots, N$, where

$$A \equiv \begin{pmatrix} \Gamma_1 & 0 & & & J_N \\ J_1 & \Gamma_2 & 0 & & \\ & J_2 & \cdots & & \\ 0 & & & J_{N-1} & \Gamma_N \end{pmatrix} \tag{7}$$

and the eigen-energy of the original Hamiltonian Eqn. (6) is given by

$$E = 2\sum_{k=1}^{N} n_k \sqrt{\lambda_k} - \sum_{k=1}^{N} \sqrt{\lambda_k}$$

where $n_k \in \{0, 1\}$ is the Fermionic occupation number. It follows that the gap between the ground state and first excited state is given by $\Delta E_{\min} \equiv \min_{k=1}^{N} \sqrt{\lambda_k}$. Since we are intereted in the gap squared in Eqn. (5), we consider the asymtoptic behaviour of $\lambda_m(N) \equiv \Delta E_{\min}^2$ with system size N, which is the minimum eigenvalue of matrix $A^T A$.

For cases when all $\Gamma_i \equiv \Gamma$ and $J_i \equiv J$, the solution is trivial

$$\lambda_m(N) = \begin{cases} (\Gamma - J)^2 & \text{N is even} \\ (\Gamma^2 + J^2 - 2\Gamma J \cos(\pi/N)) & \text{N is odd} \end{cases} \tag{8}$$

Note that there is difference between N even and odd introduced by the c-cyclic boundary, which can be problematic when N is finite since gap isn't a thermodynamic quantity[13]. However, as N goes to inifinity, the difference in Eqn. (8) and in the numerical results between N odd and even disappears, justifying this choice of boundary condition when considering the asymtoptic behaviour of λ_m with system size N.

With the disribution of the field Γ_i and couplings J_i fixed, the gap square $\lambda_m(N)$ will fluctuate among instances and obey certain distribution, we expect the distriubtion to vary with the system size N, giving a description of how the running time scales with N. However, we need to note since each system is only characterized by the distribution of parameters, this means as N grows, the sampling of new parameters will change the fluctuation property, in this sense $\lambda_m(N)$ for a specific system may not even decrease monotonically. So the scaling of $\lambda_m(N)$ with system size N is only well defined in the statistical sense, i.e. for the distribution. To characterize a distribution, we focus on the mean value $\langle \lambda_m(N) \rangle$.

IV. SCALING OF $\langle \lambda_m(N) \rangle$ WITH N

For Ising spin chain under transverse field, Fisher *et. al.*[1, 2, 10] have shown that there are three kind of scaling laws, (*i*) powerlaw in Griffith phase (*ii*) exponential law under critical condition (*iii*) non vanishing gap. While Fisher used strong bond perturbative Renormalization Group method, in Appendix C we give a simpler and more direct proof of the existence of these phases and their conditions.

A. Griffith Phase, power law

According to Fisher's result, when there is overlap of the distribution of Γ_i and J_i, i.e. $[\Gamma_{\min}, \Gamma_{\max}] \bigcap [J_{\min}, J_{\max}] \neq \emptyset$, but not on the critical line which we will discussed later, $N^z \lambda_m(N)$ obeys a distribution independent of system size N. We numerically verified the invariant distribution up to size $N = 512$(see Fig. 2(a)). This indicates a power law scaling of

$$< \lambda_m(N) > \propto N^{-z} \tag{9}$$

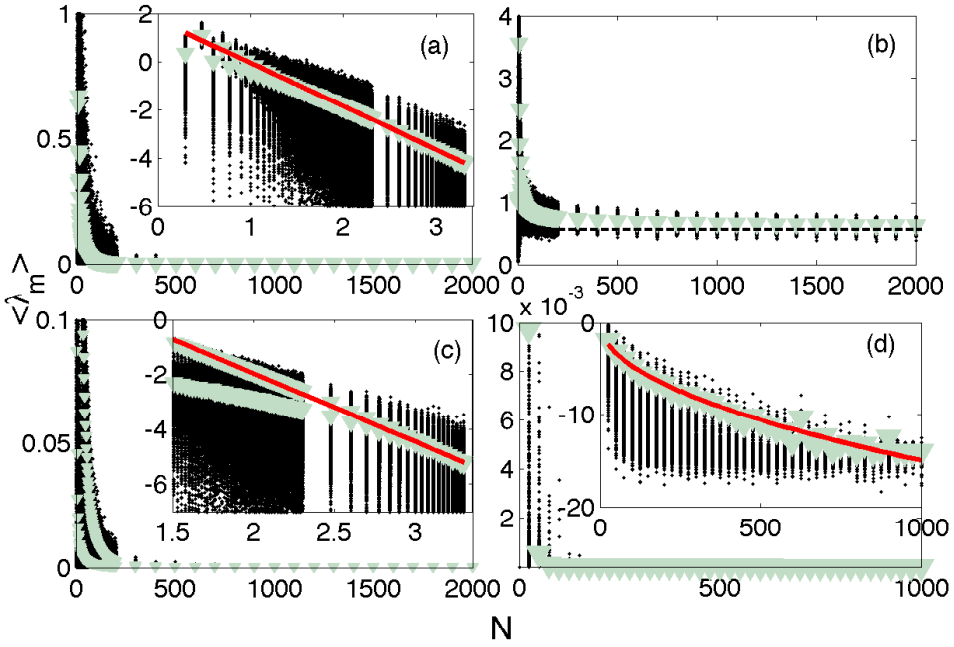

FIG. 1: In (a)-(d), the black dots are numerical data of $\lambda_m(N)$ of 1000 samples for each N; the green triangles are the mean value $\langle\lambda_m(N)\rangle$ averaged over the 1000 random samples; the subplot in (a) and (c) is the corresponding $log_{10}(\langle\lambda_m(N)\rangle)$ vs $log_{10}(N)$ plots, the subplot in (d) is the $log_{10}(\langle\lambda_m(N)\rangle)$ vs. N plot. The red lines are the fittings. (a)Griffith phase with power law vanishing gap, $\Gamma \in [4,5], J \in [1,8]$;Fitting, $y = -1.8x + 1.736$ (b)Non vanishing gap, $\Gamma \in [4,5], J \in [3,4]$; (c)Critical condition case,with ambiguious powerlaw and exponential law vanishing gap, $\Gamma \in [4,5], J \in [4,5]$;Fitting, $y = -2.495x + 3.014$; (d)Critical condition case, with exponential vanishing gap, $\Gamma \in [0,9], J \in [0,9]$;Fitting, $y = -0.4737\sqrt{x}$;

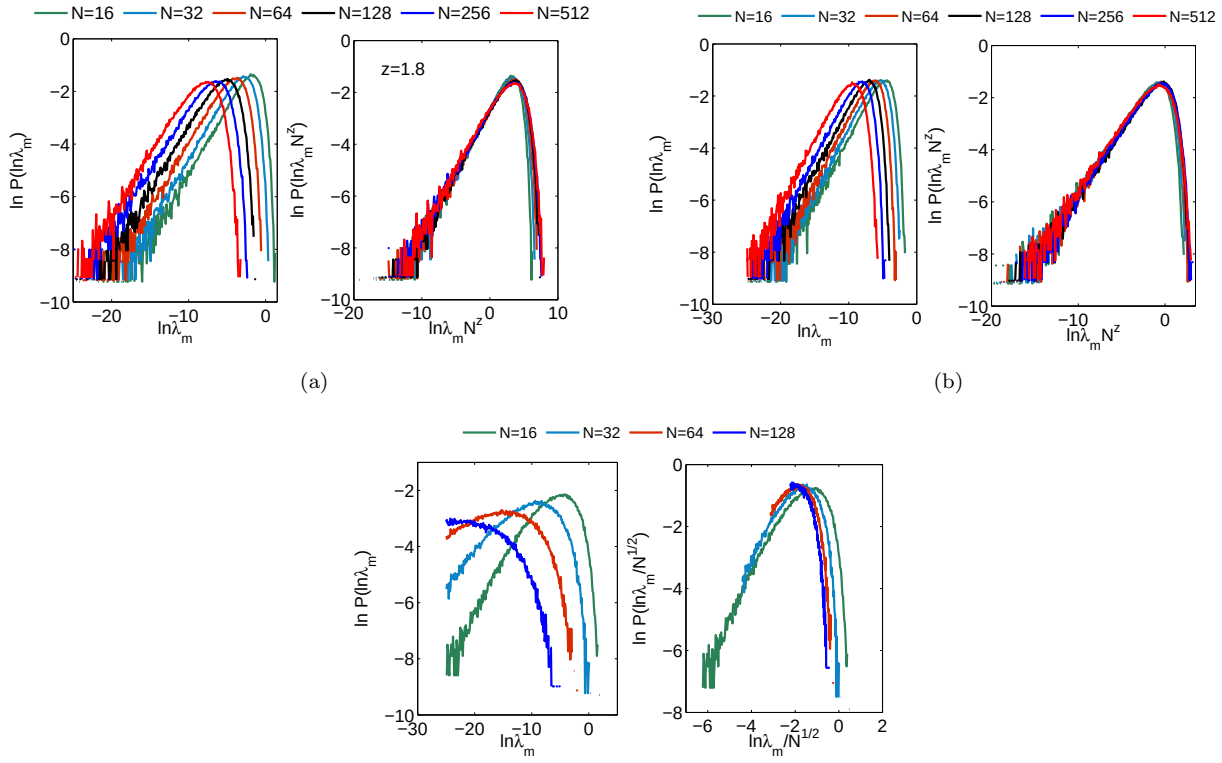

FIG. 2: (a)$\Gamma_i \in [4,5], J_i \in [1,8]$, Griffith phase, power law (b)$\Gamma_i \in [4,5], J_i \in [4,5]$, critical condition, exponential law, but with small exponents, close to powerlaw in small N. (c)$\Gamma_i \in [0,9], J_i \in [0,9]$, critical condition, exponential law. All distribution are calculated with 10^6 samples.

In the thermal dynamical limit, the gap will vanish, giving infinite phase transitions within the region. This phase is called Griffith phase[14].

In Fig. 1(a) we calculated an example of Griffith phase by choosing $\Gamma_i \in [4,5]$ and $J_i \in [1,8]$, we can see up to $N = 2000$, $\langle \lambda_m(N) \rangle$ obeys a powerlaw with exponent $z = 1.8$, and we have checked various choice of $\Gamma_{\min}, \Gamma_{\max}, J_{\min}, J_{\max}$, powerlaw with different exponents z are discovered, the dependence of z are calculated in Section. V.

However, this polynomial scaling is only for a typical case, at any N the distribution of $\ln \lambda_m(N)$ has a long tail on the negtive side, meaning the gap can be exponentially small in some rare cases. The probability, as we will see, is very small.

To get a sense of how well the $\langle \lambda_m(N) \rangle$ represents the typical cases, we calculate the probability that the gap square $\lambda_m(N)$ is larger than a lower bound set by dividing $\langle \lambda_m(N) \rangle$ with certain number α, i.e. $\mathrm{Prob}[\lambda_{\mathrm{m}}(\mathrm{N}) > \langle \lambda_{\mathrm{m}}(\mathrm{N}) \rangle / \alpha]$. For the example of $\Gamma_i \in [4,5], J_i \in [1,8]$, we list the probability in the following table

α	1	4	16	64	256	1024
Prob.	0.26	0.52	0.72	0.85	0.92	0.96

TABLE I: $\Gamma_i \in [4,5], J_i \in [1,8]$, $\mathrm{Prob}[\lambda_{\mathrm{m}}(\mathrm{N}) > \langle \lambda_{\mathrm{m}}(\mathrm{N}) \rangle / \alpha]$.

We see that by choosing certain α we obtain a polynomial scaling lower bound and a constant portion of the random cases are always larger than this bound. By choosing α properly large, nearly all random cases can be bounded by the lower bound. Consequently $\langle \lambda_m(N) \rangle$ is indeed a good measure of the overall values of $\lambda_m(N)$.

B. Critical condition, exponential law

Within the Griffith phase, where there are infinite phase transitions with powerlaw closing gaps, surprisingly along the line of

$$\langle \ln \Gamma_i \rangle = \langle \ln J_i \rangle \tag{10}$$

, the gap closes even faster. Accoding to Fisher's result[1, 2], under the critical condition $\ln \lambda_m(N)/\sqrt{N}$ obeys a distribution independent of system size N. We numerically verified the invariant distribution for the example of $\Gamma_i, J_i \in [0,9]$(see Fig. 2(c)). This indicates an exponential scaling of $\langle \lambda_m(N) \rangle$

$$\langle \lambda_m(N) \rangle \propto \exp(-a\sqrt{N}) \tag{11}$$

And we calculated corresponding $\langle \lambda_m(N) \rangle$ in Fig. 1(d), the subplot shows a good fitting of $\log_{10}(\langle \lambda_m(N) \rangle) \propto \sqrt{N}$, indicating the exponential relation. However, when we go to another case of $\Gamma_i, J_i \in [4,5]$, up to $N = 2000$, actually a very good powerlaw is observed(see Fig. 2(b), Fig. 1 (c)). This may be due to a very small exponent a in the exponential law Eqn. (11), N need to really large(of the order of $1/a^2$) for the exponential law to be dominant.

For the exponential scaling case, numerically we find that for all finite α $\mathrm{Prob}[\lambda_{\mathrm{m}}(\mathrm{N}) > \langle \lambda_{\mathrm{m}}(\mathrm{N}) \rangle / \alpha]$ goes to zero as N increases, this can be understood from Fig. 2(c). We see that as N increases, the distribution has weight more and more on the long tail of the negative $\ln \lambda_m$, while the mean value of λ_m are dominant by the largest part of $\ln \lambda_m$.

V. CALCULAION OF EXPONENT z

Note that exponent z is only well defined for the powerlaw closing gap in Griffith phase; Since (1) Fisher *et.al.*showed that the exponent will diverge close to critical condition, thus numerically it will be hard to see the difference of Griffith phase close to critical condition and critical condition; (2)the transition from griffith phase to non-vanishing gap normal phase isn't sharp; we use the same numerical procdure(taking a power law fit of 11 points in $N \in [100, 200]$) to calculate an effective powerlaw exponent z for all cases.

In this way, a close to zero z will mark the boundary of griffith phase; a large z will mark the start of critical condition; We numerically calculated the exponent z under different distribution of Γ_i, J_i. We fix $\Gamma_i \in [4.5, 4.5]$(Fig. 3 (a),(b)), $\Gamma_i \in [4,5]$(Fig. 3 (c),(d)), $\Gamma_i \in [2,7]$(Fig. 3 (e),(f)), $\Gamma_i \in [0.1, 8.9]$(Fig. 4 (a),(b)) and varies $J_{\min}, J_{\max}$ to study all possible distributions of J_i. We see that all non-zero exponent z are cases when $[\Gamma_{\min}, \Gamma_{\max}] \bigcap [J_{\min}, J_{\max}] \neq \emptyset$, this means that the criteria of griffith phase works well; We also find that approaching to limit of large $J_{\min}/J_{\max}$ ratio on the critical line, we do see a large increase on z, note that due to the way we numerically calculate, the effective z won't diverge, but a large value will indicate such a divergence.

VI. A RETHINK ON STRATEGY OF QAA

To summarize, for a restricted simple class of MAX 2-SAT problem, the conventional QAA with the transverse field Eqn. (2) as initial Hamiltonian $H(0)$, will lead to time evolution of a Hamiltonian in the form of transverse nearest neighbour Ising spin chain. Because the coupling constants J_i's distribution $[J_{\min}, J_{\max}]$ are fixed by the detiails of the MAX 2-SAT problem and a changing field is equilvalent to changing the scale of the distribution with fixed $J_{\max}/J_{\min}$, the adiabatic changing of the Hamiltonian corresponds to the straight dotted line $J_{\max}/J_{\min} = \text{constant} > 1$ in Fig. 3 (b)(d)(f) and Fig. 4(b).

Along this straight line trajectory, we see that the Hamiltonian will inevitably cross the Griffith phase region and the critical condition line, giving an exponential

scaling small gap. Thus the running time of the conventional QAA algorithn specified in the beginning need an exponential time to solve the restricted simple class of MAX 2-SAT problem. For the general MAX 2-SAT problem, the resulting Hamiltonian Eqn. (3) has long range interaction beyond nearest neighbour, and may even give more complicated phase diagrams. A straight line from Origin to Infinity will inevitably cross all the possible critical conditions. This fact poses a serious question on the ability of conventional QAA.

Luckily, QAA allows all different choices of initial Hamiltonian $H(0)$ and the trajectory of adiabatic evolution. We propose that by some clever strategy of choosing the initial Hamiltonian and trajectory, we will be able to get around the problem of exponential small gap. The strategy goes as following:

Given a CNF of the form of Eqn. (1)

- Step1.

  Randomly choose m states $|\psi_k\rangle$ of the Ising spin in the entire Hilbert space.

- Step2.

  Construct the corresponding m Hamiltonian H_k that has the state $|\psi_k\rangle$ as ground state.

- Step3.

  Construct the final Hamiltonian as in Eqn. (3).

- Step3.

  Perform m times the QAA with $H_k, k = 1, \cdots, m$ as the initial Hamiltonian and Eqn. (3) as the final Hamiltonian.

- Step4

  Pick the solution generated in the m trials of QAA that gives the maximum satisfying clauses in the CNF.

Now we explain why this strategy will work. Assume that in the allowed Hamiltonians, there are only finite phases seperated by phase transitions with exponential closing gap; Since the infinite phase transition in Griffith phase only has powerlaw closing gaps, it doesn't violate the assumption. Then with relatively large m, there will be initial states inside every phase, which means there will be initial states in the same phase with the final solution, thus the phase transitoin with exponential closing gap is avoided. However, we need to note that constructing the Hamiltonian from a given ground state may not be an easy task[15], but this only has to be done once for a given system and we can even form a database where standard initial Hamiltonian and Ground states can be chosen from, so hopefully this alternated QAA can promise a polynomial time on solving the restricted MAX 2-SAT problem.

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Appendix A: MAX 2-SAT to Ising model

For a CNF Eqn. 1 with N variables and M clauses, define N vectors of length M with each component $v_j^k = \pm 1, 0$, corresponding to the case of variable x_j being in clause k(or it's negation $\neg x_j$) or neither being in clause k. Then by definition we have M constrains

$$\sum_{j=1}^{N} |v_j^k| = 2, k = 1, \cdots, M \tag{A1}$$

The coefficients in Eqn. (3) is given[5] as

$$h_j = -\sum_k v_j^k, J_{ij} = \sum_k v_i^k v_j^k \equiv \vec{v_i} \cdot \vec{v_j} \tag{A2}$$

The restriction of no field and nearest neighbour coupling in Eqn. (6) then requires

$$\sum_k v_j^k = 0, j = 1, \cdots, N \tag{A3}$$

$$\vec{v_i} \cdot \vec{v_j} \equiv \sum_k v_i^k v_j^k = 0, i \neq j \pm 1, i, j = 1, \cdots, N \tag{A4}$$

Together Eqn. (A1) and Eqn. (A3), Eqn (A4) form $M + N + N(N-3)/2$ constrains, it should be no larger than the degree of freedom MN choosing each $v_j^k = 0, \pm 1$, we have the clause density

$$\alpha \equiv M/N > 1/2 \tag{A5}$$

Appendix B: Gauge transform of quantum Ising spin chain under transverse field

In dealing with classical Ising spins Hamiltonians

$$H = \sum J_{ij} S_i S_j$$

We have the gauge transform

$$S_i \to \sigma_i S_i$$
$$J_{ij} \to \sigma_i \sigma_j J_{ij}$$

under which the Hamiltonian is unchanged.

Here the quantum spin chain, the gauge transform should preserve the commutation relation between $(\hat{S}_i^x, \hat{S}_i^y, \hat{S}_i^z)$, so we have

$$\hat{S}_i^x \to \sigma_i^x \hat{S}_i^x$$
$$\hat{S}_i^z \to \sigma_i^z \hat{S}_i^z$$
$$\hat{S}_i^y \to \sigma_i^x \sigma_i^z \hat{S}_i^y$$
$$J_i \to \sigma_i^z \sigma_{i+1}^z J_i$$
$$\Gamma_i \to \sigma_i^x \Gamma_i$$

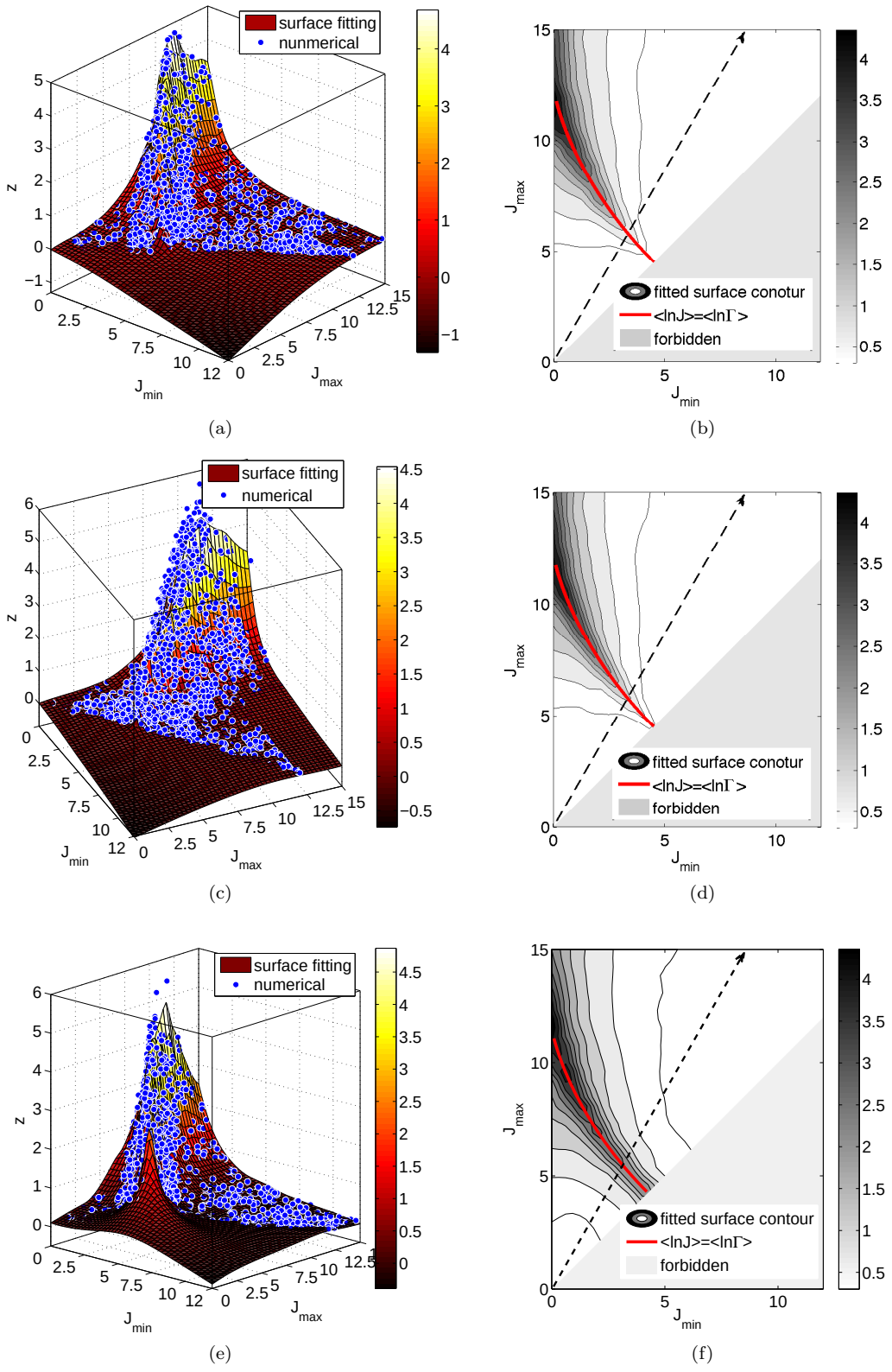
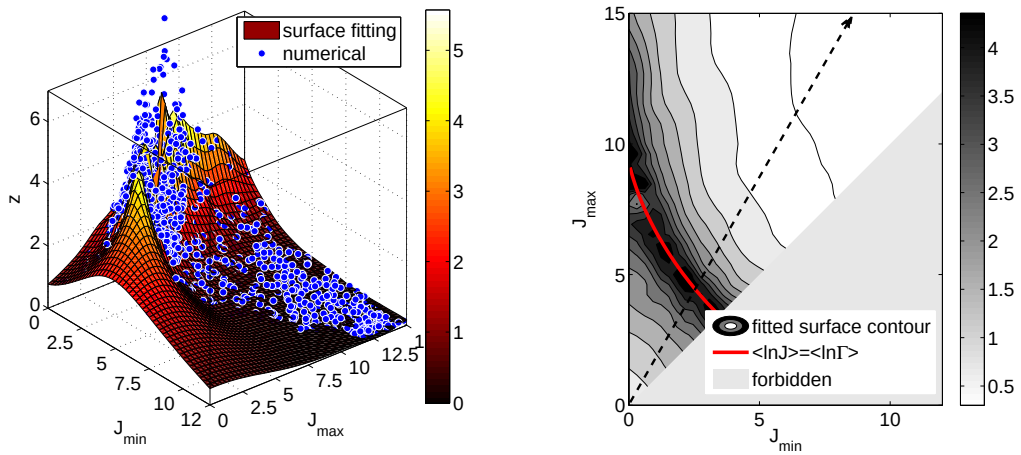


FIG. 3: (a) $\Gamma = 4.5$; the exponent z changes with the choice of $J_{\min}$ and $J_{\max}$. 1200 data points are caculated to generate the thin plate interporlation surface, note that they are choosen so that the points are more dense in area with large vairation in z to increase precession. Each data point is averaged over 500 samples. Note that since numerically it is impossible to exactly set $< \ln \Gamma >=< \ln J >$, we are not sure whether the extreme of the exponent z will diverge or not. The minimum value of $J_{\min}$ calculated is $= 0.1$. (b)the corresponding contour plot of the above fitted surface. We find that the extreme are exactly at the $< \ln \Gamma >=< \ln J >$ case.(c) $\Gamma \in [4, 5]$, 1800 data points are caculated (d)the corresponding contour plot(e) $\Gamma \in [2, 7]$; 1000 data points are caculated. (f)the corresponding contour plot



(a) (b)

FIG. 4: (a) $\Gamma \in [0.1, 8.9]$; 1000 data points are caculated (b)the corresponding contour plot

under which our Hamiltonian is unchanged, so whenever we have negative coupling, with the proper choice of $\sigma_i^x = \pm 1, \sigma_i^z = \pm 1$, all couplings can be transformed to positive, note that this transform would change the magnetization $M = \sum_i S_i^z$, but critical behaviours shouldn't change under this transform. Also this gauge transform only works with the 1D nearest neighbour coupling case(where there is no frustration), in general case, the coupling can't be made possitive in this way.

Appendix C: Proof of existence of Griffith Phase and Fisher's critical condition

If certain $J_i = 0$, then the system can be broken into two seperate system; So we consider all $J_i > 0, \Gamma_i > 0$ case. We want to proof an upper bound of the minimum gap square $\lambda_m(N)$. From spectrum theorem, the minimum eigenvalue $\lambda_m(N)$ of $A^T A$ satisfy

$$\lambda_m(N) = \mathrm{Min}\{\mathrm{x^T A^T Ax} = \mathrm{(Ax)^T (Ax)}, \mathrm{x^T x} = 1\} \quad \text{(C1)}$$

thus in order to give upper bound of $\lambda_m(N)$, it suffices to choose certain normal vector x_0 that satisfies $x_0^T x_0 = 1$ and evaluate $(Ax)^T(Ax)$. Here we choose

$$\begin{cases} Ax &= (a_1, 0, 0, \cdots, 0)^T \\ x^T x &= 1 \end{cases} \quad \text{(C2)}$$

(note that equivalently the non-zero a_1 can a non-zero a_i for other components of the vector, so together we can generate N inequalities)so that

$$\lambda_m(N) \leq (Ax)^T (Ax) = a_1^2 \quad \text{(C3)}$$

;Solve Eqn. (C2) we have

$$\begin{cases} x &= x_1(1, -\frac{J_1}{\Gamma_2}, (-)^2 \frac{J_1}{\Gamma_2}\frac{J_2}{\Gamma_3}, \cdots, (-)^{N-1}\frac{J_1}{\Gamma_2}\frac{J_2}{\Gamma_3}\cdots\frac{J_{N-1}}{\Gamma_N})^T \\ x_1^2 &= \frac{1}{1+(\frac{J_1}{\Gamma_2})^2+(\frac{J_1}{\Gamma_2}\frac{J_2}{\Gamma_3})^2+\cdots+(\frac{J_1}{\Gamma_2}\frac{J_2}{\Gamma_3}\cdots\frac{J_{N-1}}{\Gamma_N})^2} \end{cases} \quad \text{(C4)}$$

Thus

$$a_1 = \Gamma_1 x_1 + J_N x_N \quad \text{(C5)}$$

$$= \Gamma_1(1 + (-)^{N-1}\frac{J_N}{\Gamma_1}\frac{J_1}{\Gamma_2}\frac{J_2}{\Gamma_3}\cdots\frac{J_{N-1}}{\Gamma_N})x_1 \quad \text{(C6)}$$

let $y_i = J_i/\Gamma_{i+1}, i = 1, \cdots, N-1, y_N = J_N/\Gamma_1$, which are obviously iid, from Eqn. (C3) we have

$$\lambda_m(N) \leq \Gamma_1^2 \frac{(1 + (-1)^{N-1} y_1 y_2 y_3 \cdots y_N)^2}{1 + y_1^2 + (y_1 y_2)^2 + (y_1 y_2 y_3)^2 + \cdots (y_1 y_2 y_3 \cdots y_{N-1})^2} \quad \text{(C7)}$$

Note here we change from $2N$ iid variables to N iid variables, because only the relative distribution matters, the

abosulate values are set by Γ_1^2, equivalently we have

$$\frac{\lambda_m(N)}{\Gamma_1^2} \leq \frac{(1+(-1)^{N-1}y_1y_2y_3\cdots y_N)^2}{1+y_1^2+(y_1y_2)^2+(y_1y_2y_3)^2+\cdots(y_1y_2y_3\cdots y_{N-1})^2} \equiv f_1(N) \tag{C8}$$

Note that the above expression has a mathematical symmetry, devide the numerator and denominator with $(y_1\cdots y_N)^2$

$$\Longleftrightarrow \frac{\lambda_m(N)}{\Gamma_1^2} \leq y_N^{-2}\frac{(1+(-1)^{N-1}(y_1\cdots y_N)^{-1})^2}{1+y_{N-1}^{-2}+y_{N-1}^{-2}y_{N-2}^{-2}+\cdots(y_{N-1}y_{N-2}\cdots y_1)^{-2}} \tag{C9}$$

recall that $y_N = J_N/\Gamma_1$ we have

$$\Longleftrightarrow \frac{\lambda_m(N)}{J_N^2} \leq \frac{(1+(-1)^{N-1}(y_1\cdots y_N)^{-1})^2}{1+y_{N-1}^{-2}+y_{N-1}^{-2}y_{N-2}^{-2}+\cdots(y_1y_2\cdots y_{N-1})^{-2}} \tag{C10}$$

if we define $w_i = 1/y_{N-i}, i = 1,\cdots,N-1, w_N = 1/y_N$, we can arrive at a similar equaion

$$\Longleftrightarrow \frac{\lambda_m(N)}{J_N^2} \leq \frac{(1+(-1)^{N-1}w_1\cdots w_N)^2}{1+w_1^2+w_1^2w_2^2+\cdots(w_1w_2\cdots w_{N-1})^2} \tag{C11}$$

Since Γ_1, J_N are non-zero. To summarize, we have now arrived at two bounds

$$\frac{\lambda_m(N)}{\Gamma_1^2} \leq \frac{(1+(-1)^{N-1}y_1y_2y_3\cdots y_N)^2}{1+y_1^2+(y_1y_2)^2+(y_1y_2y_3)^2+\cdots(y_1y_2y_3\cdots y_{N-1})^2} \equiv f_1(N) \tag{C12}$$

$$\frac{\lambda_m(N)}{J_N^2} \leq \frac{(1+(-1)^{N-1}w_1\cdots w_N)^2}{1+w_1^2+w_1^2w_2^2+\cdots(w_1w_2\cdots w_{N-1})^2} \equiv g_1(N) \tag{C13}$$

By permutation we arrive at $2N$ upper bounds, $f_1(N),\cdots,f_N(N)$, $g_1(N),\cdots,g_N(N)$. Let's look at two of them, $f_1(N), g_{p(1)}$, where $g_{p(1)}$ is the permutation such that $w_{p(i)} = 1/y_i$, for simplicity we just omit the notation of permutation P here. We want the upper bound to go to zero as N increases. Denote $W_k = w_1w_2\cdots w_k, Y_k = y_1y_2\cdots y_k$, obviously

$$W_kY_k = 1, w_ky_k = 1 \tag{C14}$$

the bound become

$$f_1(N) = \frac{(1+(-1)^{N-1}Y_N)^2}{1+Y_1^2+Y_2^2+Y_3^2+\cdots Y_{N-1}^2} \tag{C15}$$

$$g_1(N) = \frac{(1+(-1)^{N-1}W_N)^2}{1+W_1^2+W_2^2+\cdots W_{N-1}^2} \tag{C16}$$

Since $W_NY_N = 1$, without loss of generality let's assume $Y_N < 1$,

$$\frac{1}{f_1(N)} \geq \frac{1+Y_1^2+Y_2^2+Y_3^2+\cdots Y_{N-1}^2}{4} \tag{C17}$$

under the restriction $Y_N \leq 1$.

If we consider

$$S_k = \ln Y_k = \sum_{i=1}^{k}\ln y_i \tag{C18}$$

then the above equation can be viewed as random walk, since $\ln y_i$'s are IID. The random walk is characterized by two parameters, $\mu = \langle \ln y_i\rangle$ and $\sigma = \langle(\ln y_i)^2\rangle - \langle \ln y_i\rangle^2$.

How does this random walk go to $Y_N \leq 1$(it must go to, since without loss of generality $Y_N < 1$) decides the behaviour of the bound.

1. If $E[Y_k] = E[W_k] = 1$, i.e. $\mu = 0$

In this $\mu = 0$ symmetric random walk, at any N the probability that it reaches $Y_N < 1$(i.e. $S_N = 0$) is $1/2$, so we expect the restriction at $N \to \infty$ won't influence the first $N/2$ steps, and these $N/2$ steps are already enough(consider lognormal distribution) to ensure

$$\lim_{N\to\infty} E[\frac{1}{f_1(N)}] = \infty \tag{C19}$$

2. If $E[Y_k] \neq E[W_k] \neq 1$

Then it's not symmetric random walk, the probability that $Y_N \leq 1$ is small, but there are two subcases.

- There is no overlap between the distribution of J_i and Γ_i, in this case if $Y_N \leq 1$, then $y_i = J_i/\Gamma_{i+1} < J_{\max}/\Gamma_{\min} < 1, \forall i$, then it is a "one way random walk", you can never go back. So there are strong correlation between the past and the future, at any large N the restriction that $Y_N \leq 1$ will be important.

$$\lim_{N\to\infty} Y_N = 0 \tag{C20}$$

$$\begin{aligned}\lim_{N\to\infty} 1 + Y_1^2 + Y_2^2 + \cdots + Y_{N-1}^2 \\ < \lim_{N\to\infty} \sum_{i=0}^{N-1} (J_{\max}/\Gamma_{\min})^{2i}\end{aligned}$$

$$= 1/(1 - (J_{\max}/\Gamma_{\min})^2) \tag{C21}$$

Then upper bound Eqn. (C15) doesn't go to zero, and so we can't prove a vanishing gap for this case, and it is likely that the gap will not vanish.

- There is overlap between the distribution. Then there is non zero probability that the random walk goes to $Y_N < 1$(i.e. $S_N = 0$), and still we expect the bound goes to infinity.

Note that the above process indicates that each bound that vanishes in large system size limit can give a critical situation, corresponding to one type of sufficient but not necessary phase transition condition, this indicate multiple types of phase transitions, because different situation the rate that the gap vanishes can be essentially different.

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