Percolation Model for Mobile Ad-hoc Random Networks

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We offer a model for a mobile ad-hoc random network, in which nodes that randomly located in a D dimensional space, have a distance-dependent transition probability to all other nodes. The model is examined in a simple mean-field like fashion. By treating the different paths in the network independently, the percolation phase transition is recovered. The critical behavior of the link length in the network above a critical point, as well as maximal link length below, are found in an approximated way. These results are compared to numerical simulation of the model, and show qualitative agreement. Furthermore, the model serves as a starting point to a short discussion on an Ising model network of the same type, and its critical behavior is calculated in the same way.

I. INTRODUCTION

Mobile ad-hoc random networks (MANET) are networks of wireless devices that can self-organized to exchange information without a relying on the existing of a fixed communication infrastructure[1]. The rapid growth of personal mobile devices with wireless capabilities have brought much interest in the recent decade to the designing and analyzing of communication protocols for such networks[2].

The basic principal of MANET is that if the density of stations in the network is sufficient, a particular station can rely on all other stations to serve as relay stations between itself and a remote and distanced station. Without any preexisting knowledge on the network structure, information packets percolate though the network until they reach their destination. This leads to a phase transition in the network[3]: at low densities the network is divided into clusters, and communication is restricted. Above a critical density there is one infinite cluster in the system, and communication is therefor guaranteed.

The phase transition in ad-hoc random networks was studied both numerically and analytically. Two main models are considered: percolation on a lattice and fixed radius[4, 5]. In the percolation model all stations resides on a square lattice, connected by bonds with probability p. For a two dimensional lattice on a square lattice the (bond) percolation critical probability is $p_c = 0.5$ [6]. The fixed-radius models have a deterministic and fixed transmission length, and the locations of the stations is uniformly distributed in the system (Fig. 1).

Understanding the properties of the network is important to the design of the network, and the behavior around the phase transition is of special practical importance. The critical density allows for communication with the lowest energy consumption per transmission, and is thus preferable. On the other hand, at the critical point the number of stations participating in the link is the largest and, in fact, diverges. Hence, property such as the minimal number of hops is interesting both from the engineering point of view, and behaves in a critically non-trivial way. It should be noted that while the phase transition itself has been the focus of several studies, the specific behavior around the critical point did not get as much attention.

We suggest in this paper a new model for mobile ad-hoc random networks, that is more physically reasonable and hopefully still computationally tractable. As in the radius fixed model, we consider nodes uniformly placed in a D dimensional space. However, instead of limiting the transmission to a fixed radius, with a binary transmission probability, we allow for the transmission probability to vary smoothly as the distance increases, as illustrated in Fig. 1.

In Sec. II the model is presented, and shown to exhibits a phase transition at a critical point. In Sec. III the critical behavior of the network is examined, and critical exponent for the average minimal number of hops above the critical point, and maximal transmission length below it are calculated approximately. Comparison to numerical simulations in 2 and 3 dimensions, in focus of Sec. IV. Section V is a detour from the MANET to the classical physical problem of the Ising model, where a similar model can be applied to calculate the critical exponent of physical properties (heat capacity, correlation length, etc.). Finally, conclusions and future prospects are presented in Sec VI.

II. THE MODEL

Let us consider a model of N stations, located at random positions $\mathbf{r}_i$ in a D dimensional space. The probability of transmission between points $\mathbf{r}_i$ and $\mathbf{r}_j$ is in a Gaussian shape, and given by:

$$P(i,j) = \exp[-(\mathbf{r}_i - \mathbf{r}_j)^2/l^2], \tag{1}$$

where l is an effective radius of transmission in the system. The communication in the network is via the flooding scheme, where each node that receives a message retransmits it, as long as maximal number of hopes in any given path is less than $K_{\max}$. This can be achieved, for example, by attaching a counter to each message that decreases by one with each retransmission. The probability of establishing a link between two nodes, i and j at distance d from each other is a sum of the contribution of all path of $K_{\max}$ hops or less. For simplicity, the different paths are assumed to be independent. The probability

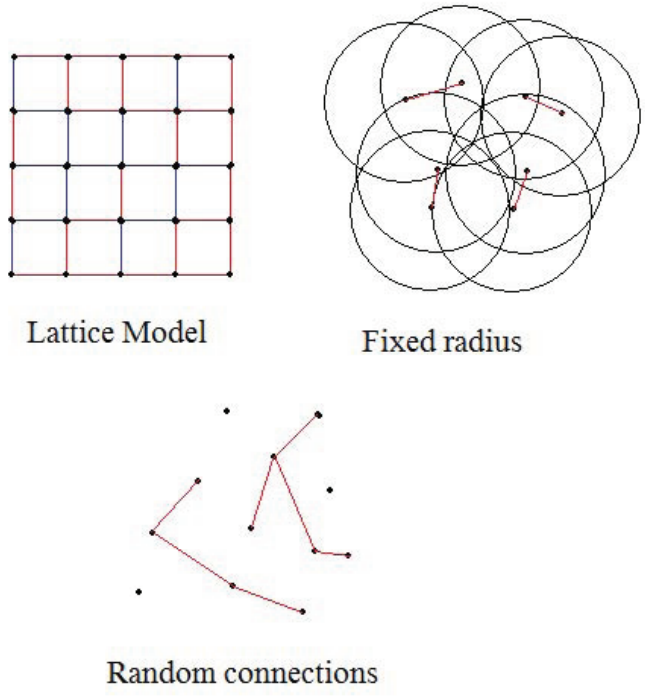

FIG. 1. Different model for ad-hoc random networks: Lattice model. Red line indicate connection between nodes. In the lattice model nodes are located on a square lattice, while in the fixed radius model they are spread in space. The model described in this paper is of a random connection of randomly placed nodes

of establishing a link through a specific path of length K is given by averaging over all possible locations of the $K-1$ intermediate station (Appex. A):

$$P(i,j) = \frac{1}{V^{K-1}} \int \mathrm{d}^D\mathbf{r}_1 \ldots \mathrm{d}^D\mathbf{r}_{K-1} \exp[-(\mathbf{r}_i - \mathbf{r}_1)^2/l^2 \ldots - (\mathbf{r}_{K-1} - \mathbf{r}_j)^2/l^2] = N^{-K} \frac{1}{(K+1)^{D/2}} \left(\frac{\sqrt{\pi} l}{a}\right)^{KD} \exp\left[-\frac{d^2}{(K+1)l^2}\right] \quad (2)$$

where $a = (V/N)^{1/D}$ is the average distance between stations. The probability that at least one of the paths of length K is successful equals to one minus the probability to fail in all of the K-long chains. The total number of K (ordered) chains is $N(N-1)...(N-K)$. Assuming $K \ll N$, so that $N!/K! \approx N^K$. Under the assumption that the chains are independent, using the approximation of the exponential function, $\exp(-a) \approx (1 - N^{-k}a)^{N^k}$, yields the following probability:

$$P_K^{(N)}(i,j) = 1 - \exp\left[-\frac{1}{(K+1)^{D/2}} \left(\frac{\sqrt{\pi} l}{a}\right)^{KD} \exp[-\frac{\mathbf{d}_{i,j}^2}{(K+1)l^2}]\right]. \quad (3)$$

The probability to establish a link that has $K_{\max}$ hops **or less** equals:

$$P_{\mathrm{link}}(K_{\max}) = 1 - \prod_{K=1}^{K_{\max}} (1 - P_K) = 1 - \exp\left\{-\sum_{K=1}^{K_{\max}} \frac{(\sqrt{\pi} l/a)^{KD}}{(K+1)^{D/2}} \exp\left[-\frac{d^2}{(K+1)l^2}\right]\right\}. \quad (4)$$

There are two different regimes in the problem: when $l > a/\sqrt{\pi} \equiv l_c$ the series in the exponent in 4 diverges as $K_{\max} \to \infty$, and $P_{\mathrm{link}} \to 1$. However, for l smaller than the critical transmission distance, the series converges and there no guarantee for establishing a link. The two regimes are illustrated in Fig 2, where Eq. **??**Pftotal) is plotted for different distances as a function of l/l_c, for the limit $K_{\max} \to \infty$.

One interesting property of the system is the distribution of the minimal value of $K_{\max}$, denoted by M, that allows for a link between points at distance d. As $d \to \infty$ this property has sense only above the critical point. Below the critical point, on the other hand, the question of the maximal transmission length arises. The critical behavior of these quantities is discussed in the next section.

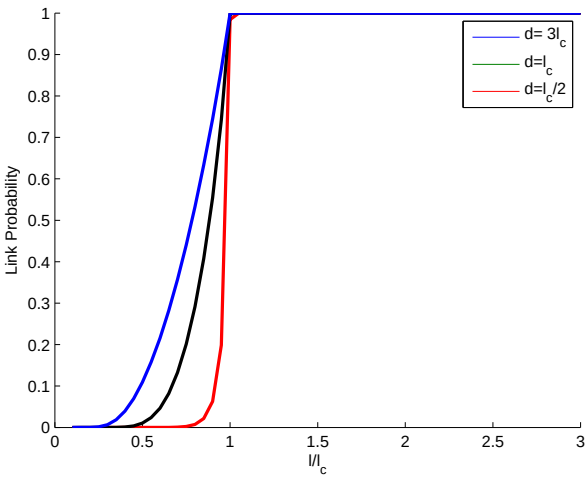

FIG. 2. The probability to establish a link between two points at distance d, according to Eq. (4), as a function of the relative transmission length l/l_c, and for different values of d.

III. CRITICAL BEHAVIOR: MINIMAL HOP AND MAXIMAL RANGE

Denoting M as the minimal $K_{\max}$ for which a link is established, the probability distribution of M equals to the probability to fail in all chains with less than M stations, multiplied by the probability to succeed in exactly

M:

$$P_{\text{minmax}}(M) = \prod_{k=1}^{M-1} (1 - P_{\text{link}}(k)) P_{\text{link}}(M), \qquad (5)$$

where P_K is given by Eq. 4. For large values of d the probability P_{minmax} can be approximated as a Gaussian distribution (detailed derivation can be found in Appx. B), with the following mean and variance:

$$\mu = \alpha - \frac{1}{2} + \frac{1}{2\beta} \log\left(2\alpha^{D/2}\beta\right)$$
$$\sigma = \frac{\sinh\beta}{2\beta^2 \exp(\beta)} \exp\left(\beta \frac{\exp(\beta)}{2\sinh\beta}\right), \qquad (6)$$

where α and β are connected to dimensionality D, average spacing a and transmission length l by:

$$\alpha = \frac{d}{l\sqrt{\beta}}$$
$$\beta = D \ln(\pi l^2/a^2)/2. \qquad (7)$$

Note that the variance does not depend on the distance, and the minimal number of hops is linear in d. In the vicinity of the critical point we can define the reduced transmission length:

$$\lambda = \frac{l - l_c}{l_c}. \qquad (8)$$

The critical behavior of the average minimal number of hops in the $d \to \infty$ limit, to leading order in λ, equals:

$$\mu = \frac{d}{l_c\sqrt{D}} \lambda^{-1/2}. \qquad (9)$$

In Fig 3 $\langle M \rangle$ is plotted as a function of λ for different distances. Both exact numerical evaluation of Eq. (5) and the approximated Gaussian version are presented and are in a good agreement for high values of d.

The second critical behavior that emerges in the system is the divergence of the maximal transmission distance when approaching the critical l from below. The probability to establish a link is given by Eq. 4, and for $K_{\max} \to \infty$ equals:

$$P_{\text{link}}(d) = 1 - \exp\left[-\sum_{k=1}^{\infty} \frac{1}{(K+1)^{D/2}} \left(\frac{\sqrt{\pi}l}{a}\right)^{KD} \exp[-\frac{d^2}{(K+1)l^2}]\right] \qquad (10)$$

The link probability has a maximum at $d = 0$, and decreases to zero. A Gaussian function can be used to approximate Eq. 10, and enables to extract the singular behavior around the critical point. Expanding the sum in the exponent of Eq. 10 to second order in d (Appx. C) yields the following approximation:

$$P_{\text{link}}(d) \approx P(0) \exp\left(-\frac{d^2}{d^2}\right), \qquad (11)$$

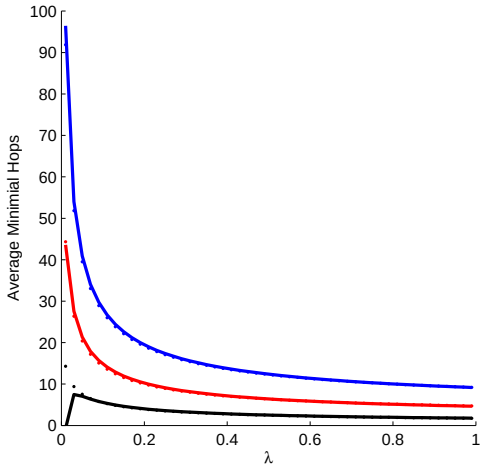

FIG. 3. The average number of minimal hops required to establish a link between two points at distance d, as a function of the reduced transmission length λ. Calculation is based both on numerical evaluation of Eq. (5) in the dotted lines, and on the approximated Eq. 6. Results plotted for $d = 2l_c$, $d = 5l_c$ and $10l_c$.

where $P(0)$ and $d_{\max}$ are parameters of the model and are summarized in Table I for different dimensionalities. In three dimensions and higher, the maximal distance remains finite as we approach the critical point, and diverges for lower dimensions. Fig. 4 illustrates the divergence of $d_{\max}$, both for exact numerical estimation of Eq. 4 and according to Table I. As $\lambda \to 0$ the Gaussian approximation is justified.

D	$d_{\max}$	$P(0)$
1	$\frac{1}{\sqrt{\zeta(3)-1}} \exp(\lambda^{-1})$	1
2	$\frac{\exp(-1/2)}{\sqrt{\zeta(2)-1)}} \lambda^{-1/2}$	1
> 2	$\frac{\exp[\zeta(D/2)-1](\zeta(D/2+1)-1)}{1-\exp[\zeta(D/2)-1]}$	$1 - \exp(-1 - \zeta(D/2+1))$

TABLE I. maximal distance as a function of dimensionality and density

In summary, the model described by Eq. (4) has several interesting features. It captures a phase transition from a network with a single infinite cluster to one with many clusters. The phase transition can be associated with either a critical *density* of stations, or with a critical *transmission length*. The critical transmission length is find to be independent of the dimension. In $2D$ the size of each cluster diverges as $|l - l_c|^{-1/2}$, and the number of hops required in a link diverges in the same way.

The main approximation in deriving Eq. (4) was to ignore the correlation between the different chains. Since different paths share same stations, this assumption is obviously wrong, and results in an exaggerated transmis-

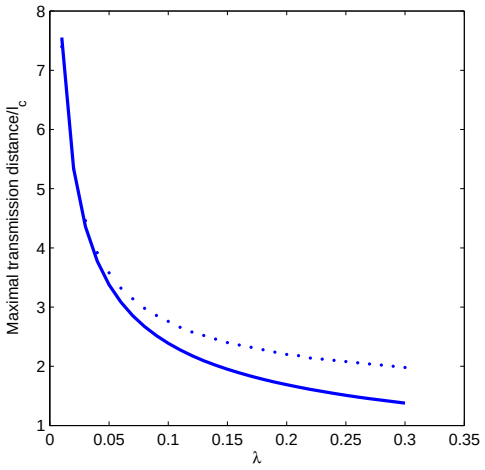

FIG. 4. Maximal transmission distance a function of λ. Calculated according to Eq. (10)(dotted line) and the approximation in Table I.

sion probability. At higher dimensions the assumption should be more reasonable, while in the low dimensions this calculation can serve as a upper bound on transmission probability. To estimate its validity, we compare in the next section our results to numerical simulations.

IV. NUMERICAL SIMULATIONS

The model presented in Sec. II can be easily studied numerically by using Monte-Carlo simulations. At each iteration, stations are randomly placed in a L^D hypercube. For simplicity, additional station is positioned at the center of the cube, and is the first to send a message. Messages propagate from station to its neighbors with probability given by Eq. 1. Each station when receives a message pass it along, while remembering the time (in number of hops) it had first received the message.

We study both 2 and 3 dimensions, for network with $N = 500$ stations. The maximal distance in the 3 dimensional case is then obviously shorter. Results are based on averaging 200 realizations. The maximal hopping number is set to be 500. It should be noted that these parameters will not allow for an exact evaluation of the behavior around the critical point: as number of hops diverges neither the size of the system nor the maximal hopping number is enough to correctly capture the behavior. Though general trends can indeed be observed.

By looking at the relative number of nodes that eventually received a message the phase transition is apparent for both dimensions(Fig. 5). The location of the critical point agrees with the calculation in 3 dimensions, but less so for the $2d$ case. In two dimensions it is difficult to decide on the exact location of the critical point, due to finite size effects of the system and the divergence of transmission length below the critical point. The criti-

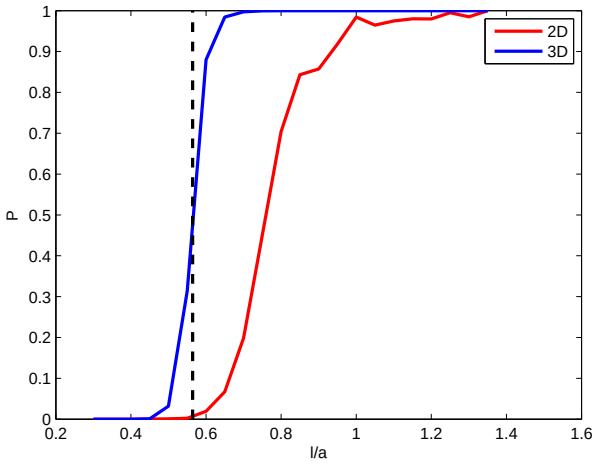

FIG. 5. Monte-Carlo simulation for the success rate at transmitting above $3a$, as a function of the relative transmission length, l/a. The black line represents the estimated critical point

cal behavior of $d_{\max}$, illustrated in Fig. 4, is plotted in Fig. 6 and show some resemblance to a divergent like behavior. However, not knowing the critical point, it is even more difficult to estimate the specific functional dependence. Finally, the average minimal number of hops,

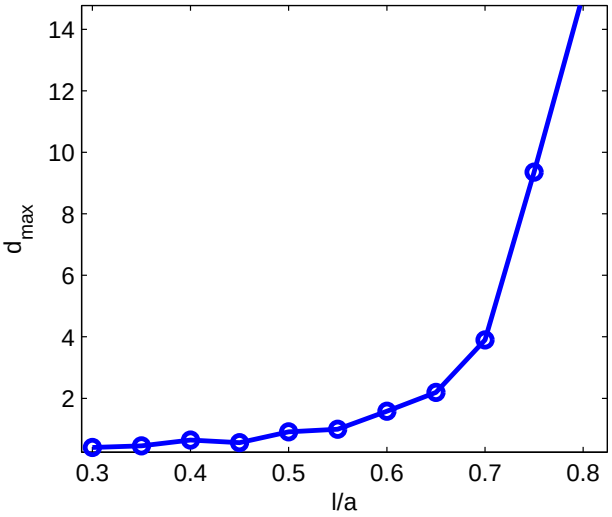

FIG. 6. Monte-Carlo simulation for $d_{\max}$ as a function of the relative transmission length, l/a. $d_{\max}$ is related to the maximal transmission length by Eq. (11).

M, is considered. On average, it should depend linearly on the distance d, and $\partial M/\partial d$ is the diverging "gain" of the system. Though not shown here, simulations suggest that there is indeed a linear dependence between

M and d. Using linear fitting, the proportionality term is extracted and plotted in Fig. 6 for different values of transmission length. Finite size effects are even of more importance in this case, as the validity our analytical model in Eq. 6 is for the large d limit.

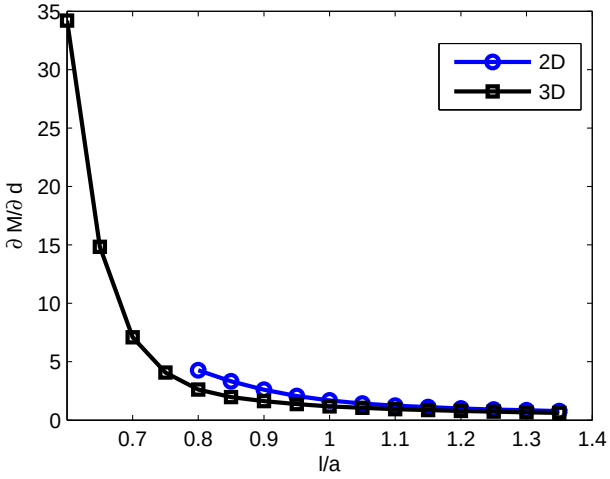

FIG. 7. Monte-Carlo simulation for the average ratio between number of hops and physical distance, as a function of the relative transmission length, l/a. $d_{\max}$ is related to the maximal transmission length by Eq. (11).

V. REVISITING THE ISING MODEL: HIGH TEMPERATURE EXPANSION

Summing over different paths in the random network was simplified by using the specific form of Gaussian shaped transmission probability. A different system in which a summation over all paths is required is the high temperature expansion of the Ising model[7]. Applying the same type of transitions in this different context yields an approximation for an Ising model with quenched disorder.

This simple generalization of the Ising model includes two modifications. First, the points are located randomly over all space, with uniform distribution. Second, interactions are not limited to nearest neighbors but instead exist between all of the spins, according to the following Hamiltonian:

$$\beta H = \sum_{i,j} J(|\mathbf{r}_i - \mathbf{r}_j|)\sigma_i\sigma_j, \tag{12}$$

where $\mathbf{r}_i$ and σ_i are the position and spin of the i^{th} particle, respectively, and $J(\mathbf{r})$ is the strength of the interaction between two spins at distance $\mathbf{r}$.

We aim to reconstruct the high temperature series expansion of the Ising model. The partition function reads:

$$\begin{aligned} Z &= \mathrm{tr}[\exp(-\beta H)] = \sum_{\{\sigma_i\}} \exp\left[\sum_{i,j} J(|\mathbf{r}_i - \mathbf{r}_j|)\sigma_i\sigma_j\right] \\ &= \sum_{\{\sigma_i\}} \prod_{i,j} [\cosh(J(|\mathbf{r}_i - \mathbf{r}_j|) + \sigma_i\sigma_j \sinh(J(|\mathbf{r}_i - \mathbf{r}_j|)] \\ &= \prod_{i,j} \cosh(J(|\mathbf{r}_i - \mathbf{r}_j|) \\ &\times \sum_{\{\sigma_i\}} \prod_{i,j} [1 + \tanh(J(|\mathbf{r}_i - \mathbf{r}_j|))\sigma_i\sigma_j]\,. \\ &= Z_0 \sum_{\{\sigma_i\}} \prod_{i,j} [1 + t_{ij}\sigma_i\sigma_j]\,, \end{aligned} \tag{13}$$

where we denote $t_{ij} = \tanh(J(|\mathbf{r}_i - \mathbf{r}_j|))$, and Z_0 is a non-singular function that does not effect the critical behavior of the model. The remains of the partition function is a sum over all possible combinations of 1 and $t_{ij}\sigma_i\sigma_j$. The only way such a combination does not average out to zero, is in the case there are even number of appearance of the i and the j points. Hence, it is a some over all possible loops, or combinations of loops. This is the same argument as in the regular high temperature expansion **??**.

A. Single loop weight

In the Ising model each loop contributes a weight of t^l, where l is the length of the loop. In the random lattice model each lattice configuration would result in a different weight, and the quenched average weight of a loop is calculated instead.

For convenience, we would like that t_{ij} to have the same form as in ad-hoc network:

$$t_{ij} = \exp[-(\mathbf{r}_i - \mathbf{r}_j)^2/2l^2], \tag{14}$$

for which the averaging over the quenched disorder is an equivalent calculation to the one in Eq. 2. However, the specific type of interaction is not particularly important, and the same calculation can be extended approximately to a much larger class of interactions. Generally, For a given loop with vertex on points $1 \ldots K$, the averaged weight equals:

$$\begin{aligned} T(K) &= \overline{t_{1,l}\prod_{i=2}^{K} t_{i-1,i1}} = \overline{\prod_{i=2}^{K} \tanh(J(|\mathbf{r}_i - \mathbf{r}_{i-1}|))} \\ &= V^{-K}\int \prod_{i=1}^{K} \mathrm{d}^D\mathbf{r}_i \tanh J(|\mathbf{r}_i - \mathbf{r}_{i+1}|), \end{aligned} \tag{15}$$

where $T(K)$ denotes the averaged loop weight, and V is the volume of the system. Since the system is invariant under translations, T can be written as an integral in Fourier space as well:

$$T(K) = V^{-K}\int \frac{\mathrm{d}^D\mathbf{q}}{(2\pi)^D} J^K(\mathbf{q}), \tag{16}$$

where $J(\mathbf{q})$ is the Fourier transform of $\tanh J(|\mathbf{r}_i - \mathbf{r}_{i+1}|)$:

$$J(\mathbf{q}) = \int \mathrm{d}^D\mathbf{r} \tanh J(|\mathbf{r}|)\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}. \quad (17)$$

Let us assume a single length scale in the problem, denoted by l. The interaction J is then written as : $J = \beta J_0 f(|\mathbf{r}|/l)$, where f is an arbitrary function, and β is the inverse temperature. The corresponding $J(\mathbf{q})$ has the form: $J(\mathbf{q}) = l^D J(\mathbf{q}l)$

$$T(K) = V^{-K}\int \mathrm{d}^D\mathbf{q}[J(\mathbf{q})]^K = \left(\frac{l^D}{V}\right)^K \int \mathrm{d}^D\mathbf{q}[J(\mathbf{q}\sigma)]^K$$
$$= \left(\frac{l^D}{V}\right)^K \int \mathrm{d}^D\mathbf{q}\mathrm{e}^{K\log([J(\mathbf{q}\sigma)])}. \quad (18)$$

Assuming that around the critical point K is large, we can expand the exponent to second order. From symmetry, there is no dependence on $\mathbf{q}$:

$$\log([J(\mathbf{q}l)]) \approx A(\beta) - B(\beta)q^2,$$
$$T(K) = \left(\frac{l^D}{V}\right)^K \int \mathrm{d}^D\mathbf{q}A^K \exp(-KBq^2)$$
$$= \left[\frac{2\pi}{KB}\right]^{D/2}\left(\frac{Al^D}{V}\right)^K. \quad (19)$$

Denoting a critical length $l_c = [V/(NA)]^{1/D}$, and a reduced length $\lambda = (l - l_c)/l_c$, the average weight has the same form as in the Gaussian model:

$$T(K) = T(0)N^{-K}K^{-D/2}(1+\lambda)^{-KD}. \quad (20)$$

The weight of a loop is then the same as if the starting point was Eq. (14).

B. Critical behavior: Free energy, Heat Capacity and correlation length

Following the standard high temperature expansion, the free energy per particle $f = \beta H/N = log(Z)/N$ equal to the sum over all possible non-intersecting loops with their corresponding weights. The number of non-intersecting loops of size K equals to the number of ways to choose K points out of N in a specific order, up to a cyclic permutation, and choice of direction: $N!/(N-l-1)!/2l$. We note that while in the regular Ising model it is difficult to count the exact number of non-intersecting loops, here the term "non-intersecting" is defined more loosely: a loop can intersect herself in real space as long as each vertex appears only once. Following Eq. TK, the singular part of the free energy reads:

$$f_{\mathrm{sing}} = \frac{1}{N}\sum_K \frac{N!}{(N-K-1)!2K}N^{-K}K^{-D/2}(1+\lambda)^{KD} \quad (21)$$

In the limit $N \to \infty$:

$$f_{\mathrm{sing}} = \frac{1}{2}\sum_K K^{-D/2-1}(1+\lambda)^{KD}. \quad (22)$$

This form of the partition function is only valid for $\lambda < 0$, and can be approximated to leading order in λ as:

$$f_{\mathrm{sing}} = \frac{1}{2}\sum_K \frac{\mathrm{e}^{-KD\lambda}}{K^{D/2+1}}. \quad (23)$$

The free energy converges for all values of λ. The heat capacity is calculated by the second derivative of the free energy:

$$C = \frac{1}{2}\sum_K \frac{\mathrm{e}^{-KD\lambda}}{K^{D/2-1}}$$
$$\approx \lambda^{D/2-1}\int_\lambda^\infty dx \frac{\mathrm{e}^{-Dx}}{x^{D/2-1}}. \quad (24)$$

To leading order in λ we get the following dependence on dimensionality:

$$C \propto \begin{cases} \lambda^{-1/2} & D = 1 \\ \lambda^0 & D = 2 \\ \lambda^{1/2} & D = 3 \\ \lambda^1 & D \geq 4 \end{cases} \quad (25)$$

Hence the only critical exponent different from zero is observed in $D = 1$.

Another property which is easy to compute is the correlation function between two spins, located at points k and k':

$$\langle\sigma_k\sigma k'\rangle = \frac{1}{Z}\mathrm{tr}[\exp(-\beta H)] =$$
$$= \frac{1}{Z}\sum_{\{\sigma_i\}}\sigma_k\sigma'_k \exp\left[\sum_{i,j} J(|\mathbf{r}_i - \mathbf{r}_j|)\sigma_i\sigma_j\right]$$
$$\frac{Z_0}{Z}\sum_{\{\sigma_i\}}\sigma_k\sigma'_k\prod_{i,j}(1+t_{ij}\sigma_i\sigma_j). \quad (26)$$

As before, the singular part of the correlation function is a sum of all combination of 1s and ts. In this case, only paths between points k and k' contribute, weighted by their probabilities along the path, $T_{k,k'}$. The weight in a path with exactly K steps equals:

$$T(d,K) = N^{-K}\frac{1}{K^{D/2}}(1+\lambda)^{KD}\exp(-d^2/Kl_c^2). \quad (27)$$

The number of path of length K connecting point k to point k' approximately N^K, where there is no ambiguity in the direction or order of these paths. Hence, the total contribution to the correlation function is:

$$\langle\sigma(0)\sigma(r)\rangle = \sum_K K^{-D/2}\exp\left[\frac{-r^2}{2Kl_c^2} + K\lambda\right] \quad (28)$$

For $D > 2$ the series converges, and at exactly the critical point the correlation decreases algebraically as r^{D-1}, while for two and one dimensions the limiting behavior is more difficult to obtain. In comparison to the known parameters of the Ising model, the above results for heat capacity and correlation length are not very satisfied.

VI. CONCLUSIONS

We constructed a mean-field like calculation to percolation in a network with quenched disorder. The network, constructed of transmitting stations in the first part, or Ising spins in the last section, was made of nodes located randomly in a D-dimensional space, and connected to all other nodes with certain probability. This simple model qualitatively captures a lot of the critical behavior of the phase transition in the network, including dependence on dimensionality, divergence of the correlation length and the minimal number of hops, etc.

Applying this specific model to an Ising ferromagnet, which had been studied extensively yields little benefit: no new insight was gained, and the critical exponents are not accurate. It is unclear though if the quenched disorder of the model has any effect, and is related to the different from the theoretical result. At least in the high dimensional limit assuming independent loops should be valid.

The mobiel network analysis, on the other hand, we believe to be of bigger importance: though the phase transition in that context is known, it was shown that the behavior of interesting properties around the critical point is non-trivial. The model presented, though inaccurate, gives an intuition that might lack in the less physical models such as the lattice percolation or fixed radius networks. A better understanding of the critical behavior can lead to a more well-designed communication systems.

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Appendix A: Link probability

A simple way to prove the link probability that has $K-1$ intermediate station, given by Eq. (2) is by induction. Let us assume that the probability for K hopes is given by Eq. 2. The probability for $K+1$ hopes equals to the probability to go from the starting station to a random station ($P_1(i,r)$), times the probability to go from that station to from that point to the destination in exactly K hopes ($P_K(r,j)$):

$$
\begin{aligned}
&P_{K+1}(i,j) = \frac{1}{V}\int d^D\mathbf{r} P_1(i,\mathbf{r}) P_K(\mathbf{r},j) \\
&= \frac{\pi^{KD/2}}{(K+1)^{D/2}}\frac{l^{KD}}{V^K}\frac{1}{V} \\
&\int d^D\mathbf{r}\exp[-\frac{(\mathbf{r}_i-\mathbf{r})^2}{l^2}-\frac{(\mathbf{r}_j-\mathbf{r})^2}{(K+1)l^2}] \\
&= \frac{\pi^{KD/2}}{(K+1)^{D/2}}\frac{l^{KD}}{V^K}\frac{l^D}{V}\exp[-\mathbf{x}_i^2-\frac{\mathbf{x}_j^2}{K+1}] \\
&\int d^D\mathbf{x}\exp[-(1+\frac{1}{K+1})\mathbf{x}^2-2\mathbf{x}\cdot(\mathbf{x}_i+\frac{\mathbf{x}_j}{K+1})] \\
&= \frac{\pi^{KD/2}}{(K+1)^{D/2}}\frac{l^{KD}}{V^K}\frac{l^D}{V}\exp\left[\frac{-(\mathbf{x}_i-\mathbf{x}_j)^2}{K+2}\right] \\
&\int d^D\mathbf{x}\exp\left\{-\left[\sqrt{1+\frac{1}{K+1}}\mathbf{x}-\frac{\mathbf{x}_i+\frac{\mathbf{x}_j}{K+1}}{\sqrt{1+1/(K+1)}}\right]^2\right\} \\
&= \frac{\pi^{KD/2}}{(K+1)^{D/2}}\frac{l^{KD}}{V^K}\frac{l^D}{V}\exp\left[\frac{-(\mathbf{x}_i-\mathbf{x}_j)^2}{K+2}\right]\left(\pi\frac{K+1}{K+2}\right)^{D/2} \\
&= \frac{\pi^{(K+1)D/2}}{(K+2)^{D/2}}\frac{l^{(K+1)D}}{V^{K+1}}\exp[-\frac{\mathbf{d}_{i,j}^2}{(K+2)l^2}]
\end{aligned}
\tag{A1}
$$

Appendix B: Approximating the average minimal link

The probability that a specific $K_{\max} = M$ would be chosen equals to the probability to fail in all chains with less than M stations time the probability to succeed in exactly M equals:

$$\begin{aligned} P_{\text{minmax}}(M) &= \prod_{k=1}^{M-1}(1-P_K)P_M \\ &= \exp(-S(M-1)) - \exp(-S(M)) \\ S(M) &= \sum_{k=1}^{M} F(K) \\ F(K) &= \frac{1}{(K+1)^{D/2}}\left(\frac{\pi l^2}{a^2}\right)^{KD/2}\exp[-\frac{d^2}{(K+1)l^2}]. \end{aligned} \tag{B1}$$

In order to get an analytical solution in this regime, let us look at the case where $d \gg l$. This is the limit where a large number of hops is required:

$$\begin{aligned} F(K) &= \frac{1}{(K+1)^{D/2}}\exp\left[-\frac{d^2}{(K+1)l^2} + \frac{KD}{2}\ln\left(\frac{\pi l^2}{a^2}\right)\right] \\ &= \frac{1}{(K+1)^{D/2}}\exp\left[\beta\frac{K^2+K-\alpha^2}{K}\right] \end{aligned} \tag{B2}$$

Where we denote $\beta = D\ln(\pi l^2/a^2)/2$, and $\alpha^2 = d^2/l^2\beta$. We can get the most of this trend by developing F around $K = \alpha$. The solutions of the equations $K^2 + K - \alpha^2$, assuming $\alpha >> 1$ are: $\pm\alpha - 1/2$:

$$\begin{aligned} F(K) &= \frac{1}{(K+1)^{D/2}}\exp\left[\beta\frac{(K-\alpha+1/2)(K+\alpha-1/2)}{K}\right] \\ &\approx \frac{\exp\left[2\beta(K+1/2-\alpha)\right]}{\alpha^{D/2}}. \end{aligned} \tag{B3}$$

And S equals:

$$\begin{aligned} S &= \sum_{k=1}^{M} F(a,d,K) \approx \frac{\exp(-2\beta(\alpha-0.5))}{\alpha^{D/2}}\sum_{k=1}^{M}\exp(2\beta K) \\ &= \gamma\exp(-2\beta(\alpha-M-0.5)), \end{aligned} \tag{B4}$$

where we denote $1/\gamma = \sqrt{\alpha}^D[1-\exp(-2\beta)]$.

To conclude, the probability of establishing a link for $K_{\max} = M$ is:

$$P_{\text{link}}(M) = 1 - \exp(\gamma\exp(-2\beta(\alpha-M-0.5))). \tag{B5}$$

The probability that $K_{\max} = M$ would be the minimal link is equal to:

$$\begin{aligned} P_{\text{minmax}}(M) = &\exp\left(-\gamma\exp[-2\beta(\alpha-M-0.5)]\right) \\ &- \exp\left(-\gamma\exp[-2\beta(\alpha-M+0.5)]\right) \end{aligned} \tag{B6}$$

The most probable minimal number of hops can be derived from (B6):

$$\begin{aligned} &\frac{\partial P_{\text{minmax}}(M)}{\partial M}|_{M=\mu} = 0 \\ &\mu = \alpha - \frac{1}{2} + \frac{1}{2\beta}\log\left(2\alpha^{D/2}\beta\right) \end{aligned} \tag{B7}$$

We can approximate the deviation from the most probable number of hops by developing the log probability to second order, and assuming a gauss model:

$$\begin{aligned} \sigma &= \frac{1}{\sqrt{(\log P_{\text{minmax}}(\mu))''}} \\ &\approx \frac{\sinh\beta}{2\beta^2\exp(\beta)}\exp\left(\beta\frac{\exp(\beta)}{2\sinh\beta}\right) \end{aligned} \tag{B8}$$

The Gaussian model works well for small values of β, however for large values we get a square shape behavior, and the approximation fail. For practical reasons, if the deviation is less than one its exact value is less important. It is interesting to see that there is no dependence on the distance, at least within in the approximation.

In summary, the minimal $K_{\max}$ is normally distributed with the following parameters:

$$\begin{aligned} \mu &= \alpha - \frac{1}{2} + \frac{1}{2\beta}\log\left(2\alpha^{D/2}\beta\right) \\ \sigma &= \frac{\sinh\beta}{2\beta^2\exp(\beta)}\exp\left(\beta\frac{\exp(\beta)}{2\sinh\beta}\right) \\ \alpha &= \frac{d}{l\sqrt{\beta}} \\ \beta &= D\ln(\pi l^2/a^2)/2 \end{aligned} \tag{B9}$$

Appendix C: Approximating $d_{\max}$

The link probability to connect two points below the critical point, can be approximated as by the following form: $P_{\text{link}}(d) = P(0)\exp\left(\frac{d^2}{d_{\max}^2}\right)$. Where we denote $d_{\max}$ as the "maximal" distance. The fact that we don't have any linear term in the distance comes form the fact that the first derivative of P_{link} with respect to d equals zero. Developing it up to second order yields:

$$\begin{aligned} P_{\text{link}} &\approx \exp\big[\ln(1-\exp(-S(0))) \\ &+\frac{1}{2}\ln(1-\exp(-S(0)))''d^2\big] \\ S(0) &= \sum_{k=1}^{\infty}\frac{1}{(K+1)^{D/2}}\left(\frac{\sqrt{\pi}l}{a}\right)^{DK} \\ &= \left(\frac{a}{\pi l^2}\right)^{D/2}\text{Li}_{D/2}\left[\left(\frac{a}{\pi l^2}\right)^{D}\right] - 1 \\ &\ln(1-\exp(-S(0)))'' = \\ &-2\exp(S(0))\frac{\sum_{k=1}^{\infty}\frac{1}{(K+1)^2}\left(\frac{\sqrt{\pi}l}{a}\right)^{2K}}{1-\exp(-S(0))} \\ &= -2\frac{\exp(S(0))}{1-\exp(S(0))} \\ &\times\left\{\left(\frac{a}{\pi l^2}\right)^{D/2}\text{Li}_{D/2+1}\left[\left(\frac{a}{\pi l^2}\right)^{D/2}\right] - 1\right\} \end{aligned} \tag{C1}$$

where $\mathrm{Li}_s(x)$ is the polylogarithm function. Looking around the critical transmission length $l \approx l_c$ we get:

$$\begin{aligned} S(0) &\approx \mathrm{Li}_{D/2}\left(\frac{n}{n_c}\right) - 1 \\ d_{\max}^2 &= \frac{\exp(S(0))}{1-\exp(S(0))}\left(\zeta(D/2+1)-1\right) \end{aligned} \tag{C2}$$

where $\mathrm{Li}_s(x) \approx \zeta(x)$ for $x = 1$ if $|s| > 1$. For $D > 2$ we get:

$$\begin{aligned} S(0) &\approx \zeta(D/2) - 1 \\ d_{\max}^{-2} &= \frac{\exp(S(0))}{1-\exp(S(0))}\left(\zeta(D/2+1)-1\right) \end{aligned} \tag{C3}$$

In the two dimensional case, $D = 2$:

$$\begin{aligned} S(0) &\approx -\log|\lambda| - 1 \\ d_{\max}^{-2} &= \exp(1)\left(\zeta(2)-1\right)\lambda \end{aligned} \tag{C4}$$

And finally in the one-dimensional case, $D = 1$:

$$\begin{aligned} S(0) &\approx \sqrt{\pi}\lambda^{1/2} - 1 \\ d_{\max}^{-2} &= \exp(1)\left(\zeta(3)-1\right)\exp(\lambda^{-1}). \end{aligned} \tag{C5}$$