Self-Organized Criticality in Anisotropic Systems

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A wide variety of systems in nature exhibit spatial and temporal scale-invariance. We review the conditions for self-organized criticality and scale-invariance in the context of nonequilibrium Langevin models. We elaborate the results of Grinstein, Sachdev and Lee [1] (GSL) who show that self-organized criticality in a system with conserved dynamics and conserved noise requires the presence of spatial anisotropy. We show that a system with conserved dynamics with identical scalings in different directions can have non-critical correlation functions under a particular condition. We show that a system with conserved dynamics with different scalings in different directions will exhibit power-law decays in the relevant directions and exponential decays in the others.

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Power-law correlations, both in space and in time, can be observed reliably in a wide variety of systems in nature [2, 3]. In time, "1/f" noise has been measured in diverse systems such as river and traffic flows, earthquakes, and stars. In space, fractal geometry has been observed in many spatially extended objects such as coastlines and rivers. How can one explain the almost universal presence of self-similar phenomena in nature?

In equilibrium systems, spatial correlations generally decay exponentially. Important exceptions are critical points, which exhibit power-law scalings with critical exponents. These cases however require fine-tuning of parameters such as the temperature, and cannot explain the ubiquitous existence of critical scalings in generic natural systems.

Bak, Tang, and Wiesenfeld have referred to the criticality exhibited by non-equilibrium, natural phenomena, which do not require a tuning parameter, as "self-organized criticality" [4]. The term "self-organized" makes reference to the fact that the system will achieve a critical state starting from any arbitrary configuration, simply through local stochastic rules.

Several models for self-organized criticality were proposed and explored starting with the original Bak, Tang and Wiesenfeld paper [4, 5]. These cellular automata models, dubbed "sandpile models", achieve a critical state starting from any configuration based on purely local interactions. "Sand" is dropped slowly onto a lattice; sites on the lattice become unstable and topple once an energy threshold (height, gradient) is reached. The study of these automata relies mostly on simulations; the calculation of critical exponents often requires mapping to a continuous model.

Largely in parallel to the development of these discrete sandpile models came the study of noisy dissipative non-equilibrium systems. These continuum models involving a field governed by a continuous Hamiltonian can be more or less mapped to the discrete sandpile models [6–8], but the link is not always clear. Noisy dissipative non-equilibrium systems can show generic scale invariance under many conditions. Here, we consider an example of such a system governed by a Langevin equation and study the conditions in which generic scale invariance and power-law correlations will emerge.

We first consider the simplest linear anisotropic Langevin model, with different components in the directions parallel ($x_\parallel$) and perpendicular ($x_\perp$) to the flow. This model could correspond, for example, to the flow of water along a river. Using the notation in reference [9], a the fluctuations of a field $h(x,t)$ will be governed by the noisy diffusion equation

$$\frac{\partial h(x,t)}{\partial t} = v_\parallel \partial_\parallel^2 h + v_\perp \nabla_\perp^2 h + \eta(x,t), \tag{1}$$

where $v_\parallel$ and $v_\perp$ are diffusion constants in the parallel and perpendicular directions, and the Gaussian random variable η satisfies

$$\langle \eta(x,t) \rangle = 0, \tag{2}$$

and can be chosen either to conserve h on average,

$$\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t'), \tag{3}$$

or to conserve h strictly,

$$\langle \eta(x,t)\eta(x',t') \rangle = 2(D_\parallel \partial_\parallel + D_\perp \nabla_\perp)\,\delta(x-x')\delta(t-t'). \tag{4}$$

The components in the diffusion equation are the simplest terms that preserve the translational symmetry required by the system, thus excluding for instance terms that are proportional to h.

In both cases, we would like to calculate the correlation function $G(x) \equiv \langle h(x,t)h(0,t) \rangle$ which is the inverse Fourier transform of the susceptibility $\chi(q) \equiv \lim_{t\to\infty} \langle |h(q,t)|^2 \rangle$. Taking the Fourier transform of equation (1), the Fourier modes evolve according to

$$\frac{\partial h(q,t)}{\partial t} = v_\parallel q_\parallel^2 h + v_\perp q_\perp^2 h + \eta(q,t) = \frac{h(q,t)}{\tau(q)} + \eta(q,t), \tag{5}$$

with characteristic times

$$\tau(q) = \frac{1}{v_\parallel q_\parallel^2 + v_\perp q_\perp^2}. \tag{6}$$

The evolution of each mode follows

$$h(q,t) = h(q,0)e^{-t/\tau(q)} + \int_0^t \mathrm{d}t' e^{-(t-t')/\tau(q)} \eta(q,t), \quad (7)$$

and $\chi(q)$ is equal to

$$\lim_{t\to\infty} \langle |h(q,t)|^2 \rangle = \frac{D}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2}, \quad (8)$$

in the case where noise is conservative on average (equation (3)) and to

$$\lim_{t\to\infty} \langle |h(q,t)|^2 \rangle = \frac{D_{\|} q_{\|}^2 + D_{\perp} q_{\perp}^2}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2}, \quad (9)$$

in the case where noise is strictly conservative (equation (4)). The correlation functions for both cases can be found by taking an inverse Fourier transform. We evaluate the Fourier transform for the non-conservative case first:

$$G(x) = \langle h(x,t)h(0,t) \rangle = \int \mathrm{d}^{d-1} q_{\perp} \mathrm{d}q_{\|} e^{i(q_{\perp}x_{\perp} + q_{\|}x_{\|})} \frac{D}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2} \quad (10)$$

We first evaluate the $q_{\|}$ integral using Cauchy's residue theorem. Noting that the denominator can be separated into partial fractions,

$$\frac{1}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2} = \frac{1}{v_{\|}} \frac{1}{2iq_0} \left(\frac{1}{q_{\|} - iq_0} - \frac{1}{q_{\|} + iq_0} \right), \quad (11)$$

where $q_0 = \sqrt{v_{\perp}/v_{\|}} |q_{\perp}|$, we evaluate the residue in the upper half-plane at $q_{\|} = iq_0$, and

$$\int \mathrm{d}q_{\|} \frac{De^{i(q_{\|}x_{\|})}}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2} = \frac{D\pi}{\sqrt{v_{\|} v_{\perp}}} \frac{e^{-\sqrt{v_{\perp}/v_{\|}}\ |q_{\perp}| x_{\|}}}{|q_{\perp}|}. \quad (12)$$

Thus,

$$G(x) \propto D \int \mathrm{d}^{d-1} q_{\perp} \frac{e^{iq_{\perp}(x_{\perp} + i\sqrt{v_{\perp}/v_{\|}} x_{\|})}}{|q_{\perp}|}, \quad (13)$$

$$G(x) \propto D(x_{\perp} + i\sqrt{v_{\perp}/v_{\|}}\ x_{\|})^{1-(d-1)}, \quad (14)$$

and

$$G(x) \propto D(v_{\|} x_{\perp}^2 + v_{\perp} x_{\|}^2)^{\frac{2-d}{2}}. \quad (15)$$

If the noise is conservative on average, correlations will always have power-law decays regardless of the values of $v_{\|}$ and $v_{\perp}$. If the noise is strictly conservative, we evaluate the Fourier transform using the same trick and separate $\chi(q)$ into partial fractions:

$$\frac{D_{\|} q_{\|}^2 + D_{\perp} q_{\perp}^2}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2} = \frac{D_{\|}}{v_{\|}} - v_{\perp} \left(\frac{D_{\|}}{v_{\|}} - \frac{D_{\perp}}{v_{\perp}} \right) \frac{q_{\perp}^2}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2}, \quad (16)$$

and

$$\frac{q_{\perp}^2}{v_{\|} q_{\|}^2 + v_{\perp} q_{\perp}^2} = \frac{|q_{\perp}|}{2i\sqrt{v_{\perp}/v_{\|}}} \left(\frac{1}{q_{\|} - iq_0} - \frac{1}{q_{\|} + iq_0} \right), \quad (17)$$

where $q_0 = \sqrt{v_{\perp}/v_{\|}} |q_{\perp}|$ as before. Following the same steps as before (evaluation of the residue in the upper half-plane), we obtain

$$G(x) \propto \left(\frac{D_{\|}}{v_{\|}} - \frac{D_{\perp}}{v_{\perp}} \right) \int \mathrm{d}^{d-1} q_{\perp} |q_{\perp}| e^{iq_{\perp}(x_{\perp} + i\sqrt{v_{\perp}/v_{\|}} x_{\|})}, \quad (18)$$

and

$$G(x) \propto \left(\frac{D_{\|}}{v_{\|}} - \frac{D_{\perp}}{v_{\perp}} \right) (v_{\|} x_{\perp}^2 + v_{\perp} x_{\|}^2)^{-d/2}. \quad (19)$$

The correlations are again scale invariant, as long as $D_{\perp}/v_{\perp} \neq D_{\|}/v_{\|}$. If $D_{\perp}/v_{\perp} = D_{\|}/v_{\|}$, $\chi(q)$ is reduced to (equation (16)) $D_{\|}/v_{\|}$ and $G(x)$ is a delta-function.

The comparison of these two results for the simple anisotropic noisy diffusion equation illustrates a key insight into self-organized criticality originally made by Grinstein, Sachdev and Lee [1]. In strictly conservative systems, a necessary condition for self-organized criticality is the presence of a spatial anisotropy. In systems that are conservative on average, self-organized criticality will occur even in an isotropic system.

The preceding results are derived from the simple anisotropic diffusion equation, with second-order dependences in both parallel and perpendicular directions. However, many materials and phenomena, such as liquid crystals, behave differently along different directions. What kind of correlations occur if there are not simply different diffusion constants in different directions but also different diffusive scalings? We consider the following diffusion equation, with a quartic diffusion term in the perpendicular direction:

$$\frac{\partial h(x,t)}{\partial t} = v_\parallel \partial_\parallel^2 h + L_\perp \nabla_\perp^4 h + \eta(x,t). \tag{20}$$

In the conservative case, the corresponding susceptibility is (following the steps we performed before):

$$\chi(q) = \frac{D_\parallel q_\parallel^2 + D_\perp q_\perp^2}{v_\parallel q_\parallel^2 + L_\perp q_\perp^4}. \tag{21}$$

Separating into partial fractions,

$$\chi(q) = D_\parallel + \frac{D_\parallel L_\perp q_\perp^2 - D_\perp v_\parallel}{2i\sqrt{v_\parallel L_\perp}} \left(\frac{1}{q_\parallel - iq_1^2} - \frac{1}{q_\parallel + iq_1^2} \right), \tag{22}$$

where $q_1^2 = \sqrt{L_\perp / v_\parallel} q_\perp^2$. Using the Cauchy residue theorem, the integral over $q_\parallel$ is $\pi/\sqrt{v_\parallel L_\perp}(L_\perp q_\perp^2 - D_\perp v_\parallel)e^{-q_1^2 x_\parallel}$. The Fourier transform thus evaluates to

$$G(x) \propto \int \mathrm{d}^{d-1} q_\perp \left(L_\perp q_\perp^2 - D_\perp v_\parallel \right) e^{iq_\perp x_\perp - \sqrt{L_\perp / v_\parallel} x_\parallel q_\perp^2}. \tag{23}$$

This is the sum of two (d-1)-dimensional Gaussian integrals, which can be readily evaluated. The final result is:

$$G(x) \propto \sqrt{\frac{(4\pi)^{d-1}}{\left(\sqrt{\frac{L_\perp}{v_\parallel}}\, x_\parallel\right)^{d-1}}} e^{x_\perp^2 / \left(4x_\parallel \sqrt{L_\perp / v_\parallel}\right)} \left[-D_\perp v_\parallel + \frac{\sqrt{L_\perp v_\parallel}}{2x_\parallel} - \frac{v_\parallel x_\perp^2}{4x_\parallel^2} \right] \tag{24}$$

The correlations follow a power-law decay in the parallel direction and decay exponentially in the perpendicular directions. The result is not all too surprising, given that by fixing the dynamics in one direction to be of a different order than in the other directions, we have introduced a natural length scale $\sqrt{L_\perp / v_\parallel}$. Still, since the system is intrinsically anisotropic, there will always be self-organized criticality in at least one direction, as predicted by GSL, and there is no set of parameters in which $G(x)$ will simply be a delta function. Only if there is a quartic diffusion term in both directions will it be possible for the terms to cancel each other out.

There are other possible ways of adding a q^4 term, such as $\chi(q) = \frac{D_\parallel q_\parallel^2 + D_\perp q_\perp^2}{v_\parallel q_\parallel^4 + L_\perp q_\perp^2}$, with a quartic power in the parallel direction instead of the perpendicular direction. When we evaluate the $q_\parallel$ integral, there are four residues, two of which are in the upper half-plane, corresponding to the fourth roots of -1. The integral is much harder to evaluate, and there is a $\sqrt{q_\perp}$ term in the exponential instead of a $q_\perp^2$ term. Still, the results should be similar to the calculation we have made. Only if we add a nonlinear term such as ∇h^2 or ∇h^4 will the results be seriously different.

In summary, we have elaborated the results of GSL and shown in detail the calculation of the correlation function for conservative and non-conservative noise for a system following the simple anisotropic noisy diffusion equation. In addition, we showed that in the case of different orders of diffusion in different directions, the correlation function shows both power-law and exponential decays.

The thesis of GSL regarding the necessity of conservation laws for self-organized criticality has been widely discussed in subsequent literature. Several non-conservative models have been proposed to exhibit self-organized criticality, thus disproving GSL's assertion [10, 11]; however the existence of true self-organized criticality has subsequently been questioned or refuted[12, 13].

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