# Rare events in Brownian motion: a path integral approach

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In this paper I consider rare events in Brownian motion using path integral formulation, and obtain some interesting results recently found by Yang and Govern in [1], [2]. In particular, I show that the conditioned mean first passage time (CMFPT) for a 1D brownian motion to go uphill in potential landscape is equal to that to go downhill using path integral, in agreement with the results found by Yang and Govern.

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## INTRODUCTION

Consider a particle doing Brownian motion under some potential field. It is obvious that the probability for it to start from some high-energy point and end at some low-energy point after certain time is much higher than the opposite, namely to start from some low-energy point and climb up to some high-energy point. In fact by detailed balance, the ratio of the two probability should be given by

$$\frac{\mathcal{P}(0\to 1)}{\mathcal{P}(1\to 0)} = \exp(-(V(1)-V(0))/k_BT). \tag{1}$$

What sounds more interesting is that, observed and derived by Yang and Govern recently in [1], [2], if the rare event ever happens, it does so in a time that's as short as the more likely event. To put it in a more accurate form, the average time the particle needs to go from $x = 0$ and hit $x = 1$ for the first time if it does so before going back to $x = 0$, named as the conditioned mean first passage time (CMFPT) in [1], is exactly the same as that from $x = 1$ to $x = 0$, no matter what the potential looks like in between.

This result was proved in [1] using Ito calculus and Girsanove theorem. Here I show it using the formulation of path integral, which is more intuitive and friendly to many people including myself.

## BROWNIAN MOTION AND PATH INTEGRAL

The path integral representation of a brownian motion takes the form ([4])

$$\begin{aligned}
\mathcal{P}(x,\tau) &= \int \mathcal{D}x(t)e^{-S[x(t)]},\\
S[x(t)] &= \int_0^\tau dt\frac{(x'-v(x))^2}{4D}.
\end{aligned} \tag{2}$$

It is tempting to rewrite the cross term of the action as $\sim -\int dtv(x)x'dt = -\int dxv(x) = V(x(\tau)) - V(x(0))$, where $v(x) = -dV/dx$. However, this is not quite right due to fluctuation correction. The non-triviality of such terms is usually noticed in Ito integral formulation, where we have ([3])

$$\begin{aligned}
\int dx\frac{dV}{dx} &\sim \sum \delta V - \frac{1}{2}\sum\frac{d^2V}{dx^2}(\delta x)^2\\
&\sim \Delta V - D\int dt\frac{d^2V}{dx^2}.
\end{aligned} \tag{3}$$

Therefore it is more sensible to rewrite the action as

$$\begin{aligned}
S[x(t)] &= \int_0^\tau dt\frac{1}{4D}\left(x'^2+\left(\frac{dV}{dx}\right)^2\right)-\frac{1}{2}\int_0^\tau dt\frac{d^2V}{dx^2}\\
&\quad+\frac{V(x(\tau))-V(x(0))}{2D}.
\end{aligned} \tag{4}$$

The term $\int dt(d^2V/dx^2)$ can be generated in the path integral by first discretizing the time, then doing coarse-graining by integrating out $x(t_i)$ for some of the $t_i$'s. The above action turns out to be the "fixed point" of the coarse-graining process.

The conditioned probability distribution can be represented easily in the path integral formulation. For example, the probability for a particle started from $x(0) = 0$ and reach $x(\tau) = 1$ while staying within $(0, 1)$ at any time within $(0, \tau)$, is written as

$$\mathcal{P}'(x,\tau) = \int' \mathcal{D}x(t)e^{-S[x(t)]}. \tag{5}$$

Here the restricted path integral $\int' \mathcal{D}x(t)$ simply means only paths satisfying the conditions listed above are to be summed over. Technically it is not easy to calculate such restricted integrals, but various insights can be obtained from this representation.

## CONDITIONED MEAN FIRST PASSAGE TIME

The conditioned mean first passage time (CMFPT) of the particle from $x = 0$ to $x = 1$ is

$$\begin{aligned}
\tau(0\to 1) &= \frac{\int_0^\infty d\tau\mathcal{P}'(x(\tau)=1,x(0)=0)\tau}{\int_0^\infty d\tau\mathcal{P}'(x(\tau)=1,x(0)=0)}\\
&= \frac{\int' \mathcal{D}x(t)e^{-S[x(t)]}\tau}{\int' \mathcal{D}x(t)e^{-S[x(t)]}}.
\end{aligned} \tag{6}$$

Here the restricted path integral $\int' \mathcal{D}x(t)$ sums over all paths that start from $x = 0$, ends at $x = 1$ and stays within $(0, 1)$ throughout. Similarly the CMFPT from $x = 1$ to $x = 0$ is

$$\tau(1 \to 0) = \frac{\int'' \mathcal{D}x(t) e^{-S[x(t)]} \tau}{\int'' \mathcal{D}x(t) e^{-S[x(t)]}}, \tag{7}$$

where the restricted path integral $\int'' \mathcal{D}x(t)$ sums over all path that start from $x = 1$, ends at $x = 0$ and stays within $(0, 1)$ throughout.

I now show that $\tau(0 \to 1) = \tau(1 \to 0)$. This is done by matching paths in the two cases. For a path $x(t)$ such that $x(0) = 0$ and $x(\tau) = 1$, we define

$$\tilde{x}(t) = x(\tau - t). \tag{8}$$

It is straightforward to show that $S[x(t)] = S[\tilde{x}(t)] + (V(1) - V(0))/D$. Hence we have

$$\begin{aligned} \tau(1 \to 0) &= \frac{\int'' \mathcal{D}x(t) e^{-S[x(t)]} \tau}{\int'' \mathcal{D}x(t) e^{-S[x(t)]}} \\ &= \frac{\int' \mathcal{D}x(t) e^{-S[x(t)]} e^{(V(1)-V(0))/D} \tau}{\int' \mathcal{D}x(t) e^{-S[x(t)]} e^{(V(1)-V(0))/D}} \\ &= \frac{\int' \mathcal{D}x(t) e^{-S[x(t)]} \tau}{\int' \mathcal{D}x(t) e^{-S[x(t)]}} \\ &= \tau(0 \to 1). \end{aligned} \tag{9}$$

Although my derivation is carried out for 1D Brownian motion here, it clearly generalize to higher dimensions: with $D(\mathbf{x}) = D$ and $\mathbf{v}(\mathbf{x}) = -\nabla V(\mathbf{x})$, the CMFPT of a particle to go from a constant potential surface A to another constant potential surface B is equal to that from B to A, provided that both A and B are bounded (so that the time is actually finite).

I now comment on the term proportional to $\int dt (d^2V/dx^2)$ in the action (4). Without this term, the above argument can be carried out when the potential is reversed $(V(x) \to -V(x))$, and we would have concluded that the CMFPT of the particle to go from $x = 0$ to $x = 1$ is the same whether we have $V(x)$ or $-V(x)$ as the potential. However, in the presence of the $\int dt (d^2V/dx^2)$ term, such an argument would work only if $d^2V/dx^2 = 0$. This is also in agreement with the theoretical derivation and numerical simulation in Yang and Govern.

## CONCLUSION AND IMPLICATION

To summarize, I derived results obtained by Yang and Govern on rare events in Brownian motion using the formulation of path integral. In particular, I showed that the conditioned mean first passage times (CMFPT) for a particle to go uphill and downhill in potential landscape are the same. This is in agreement with the numerical simulations done in [1], [2]. It is also hopeful to observe this experimentally, for example in certain biological systems ([1]).

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