Directed Percolation on a 2D Triangular Lattice

Ruichao Ma*
Department of Physics, Harvard University
(Dated: May 10, 2010)

We study directed percolation on a 2D triangular lattice where the forward direction is perpendicular to one of the lattice axes. In addition to the geometrical description, an equivalent dynamic description of the model is presented. Simulations are performed based on an algorithm from the dynamic description to obtain the critical probability of bond occupation as $p_c \approx 0.522$. A mean field theory is used to calculate the critical exponents of the transition; and the upper critical dimension is determined by the Ginzburg criterion. We also developed a position-space renormalization group scheme to investigate the critical point.

I. INTRODUCTION

In percolation models[1, 2], bonds on a lattice are independently occupied with a probability p. For small values of p, clusters (sites connected by occupied bonds) remain of finite sizes and the system is in the isolating phase. As p is increased, the system undergoes a classical phase transition into a percolating (or conducting) phase where infinite clusters exist. In the case of directed percolation (DP)[3], the bonds are uni-directional, and sites can only be connected in a fixed direction defined globally. DP problems are common in nature, consider for example penetration of water in a porous medium under gravity, or hoping of electrons in certain solid state materials where an electric field is present. Compared to the isotropic percolation problem where there is rotational symmetry, the special 'forward' direction in DP makes the behaviour near phase transition different. Contrary to the isotropic case, DP models has not yet been solved analytically in most cases. It is also believed that the critical exponents in DP are irrational numbers[4].

If we think of the special direction in DP as 'time' axis, then a usual d-dimensional DP problem can be viewed as a time evolution problem in $(d-1)$ spatial dimensions. As a function of time, the connected clusters may grow or shrink in spatial size, depending on the bond occupation probability. This dynamic description finds itself in spreading of epidemics[5], forest fire[6], etc. While being equivalent to it's original geometrical description, the dynamic description of DP is usually better suited for computer simulations.

In the present paper, we study directed bond percolation on a 2-dimensional triangular lattice. We first formulate the model system, giving both the conventional and dynamic description (Sec. II). Next we study the critical exponents of DP phase transition in mean field approximation (Sec. III). Then in Sec. III, we perform a position space renormalization calculation to study the DP critical point.

* rma@physics.harvard.edu

II. TRIANGULAR LATTICE PROBLEM

For DP on a 2D triangular lattice, two distinct situations arise due to the relative orientation of the lattice with respect to the forward direction. They are shown in Fig. 1: (A) No bonds are perpendicular to the forward direction, each bond has a unique direction. (B) One of the lattice axes is perpendicular to the forward direction, and how these transverse bonds are occupied need to be defined. For simplicity, we let them to have probability $p/2$ to be occupied and pointing in each of the two directions along the transverse axis (up and down in Fig. 1(B)) and probability of $(1-p)$ to be unoccupied.

In the rest of the paper, we mainly focus on the case (B). This case can be thought of as DP on square lattice with a diagonal forward direction, plus additional connectivity in the transverse direction. All bonds that are not perpendicular to the forward direction are referred to as 'longitudinal' later in the text.

We now describe the dynamic interpretation for our triangular DP problem. At time t_0, we have a configuration where some sites are active (i.e. the site belongs to a connected cluster at $t < t_0$), others inactive. A site at the next time step (t_0+1) is active only if it is connected to other active sites via occupied directed bonds. In our geometry, this can happen in 2 ways: via longitudinal bonds that connect t_0 to t_0+1, or through transverse

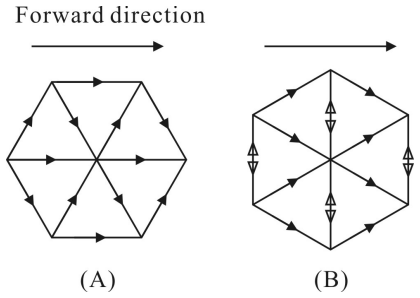

FIG. 1. Two different geometries for the DP problem on 2D triangular lattice. The case (B) can be viewed as DP on square lattice (solid arrows) with additional bonds in the transverse direction (white arrows).

bonds that connect neighbouring sites at the same time. The situation is shown in Fig. 2.

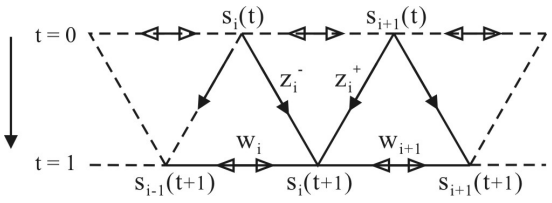

FIG. 2. Dynamic description of triangular DP model.

We generate 3 sets of random numbers ($z_i^{\pm}$, w_i) in the range $[0,1]$ for bonds in the 3 lattice axes. They are used to decide whether a bond is occupied. Start with configuration $s_i(t)$ ($s_i = 1$ for active, 0 for inactive). The new configuration $s_i(t+1)$ is determined in 2 steps:

First consider longitudinal bonds. An intermediate configuration is obtained:

$$\sigma_i(t+1) = \begin{cases} 1 \text{ if } & s_i(t) = 1 \ \& \ z_i^- < p \\ \text{ or } & s_{i+1}(t) = 1 \ \& \ z_i^+ < p \\ 0 \text{ if } & \text{otherwise} \end{cases} \qquad (1)$$

Then updating the transverse bonds to get:

$$s_i(t+1) = \begin{cases} 1 \text{ if } & \sigma_i(t+1) = 1 \\ \text{ or } & \sigma_{i-1}(t+1) = 1 \ \& \ w_i < p/2 \\ \text{ or } & \sigma_{i+1}(t+1) = 1 \ \& \ p/2 \le w_{i+1} < p \\ 0 \text{ if } & \text{otherwise} \end{cases} \qquad (2)$$

A simulation is carried out with the above algorithm, with a random initial configuration. The resulting time evolutions are shown in Fig. 3 for 3 different values of p. For small $p < p_c$, the number of active sites decreases to zero, corresponding to the isolating phase. For $p > p_c$, the number of active sites grows to a saturated value, giving rise to the percolating phase. Near criticality, the number of active sites remains about constant. The spreading behaviour is insensitive to initial conditions, and the simulations give a critical value of $p_c \approx 0.522$.

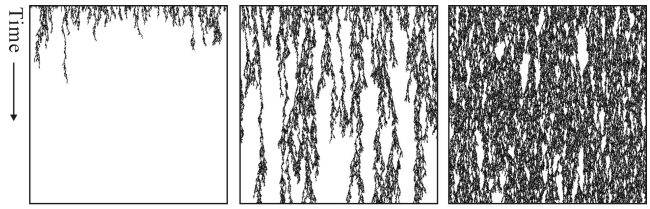

FIG. 3. Simulation of DP on a triangular lattice of size 1024×1024, using the dynamic description and algorithm in Eqn. 1 and Eqn. 2. The initial state (top row) is a random configuration with 30 percent occupation.

III. MEAN FIELD TREATMENT

The mean field theory results in this section are independent of the microscopic geometry of the model under consideration.

Consider first the case of isotropic percolation. Define an order parameter $P = P(\mathbf{x})$ as the probability that a site near $\mathbf{x}$ belongs to an infinite cluster. The case $P = 0$ denotes the isolating phase where only finite clusters are formed, whereas $P \neq 0$ is the percolating (or conducting) phase where infinitely separated sites can be connected. We can write down a Landau-Ginzburg Hamiltonian in terms of this order parameter [7]:

$$\beta H_{iso} \propto \int d^d\mathbf{x} \Psi[P]$$

$$\Psi[P] = \frac{K}{2}(\nabla P)^2 + (p_c - p)P^2 + bP^3 + \ldots \qquad (3)$$

where p_c is the critical probability, K and b are positive parameters. Terms linear in P do not appear because they can be absorbed into the quadratic terms. We assume higher order terms in the Hamiltonian provides the required stability. In the saddle point approximation ($\nabla P = 0$), a phase transition occurs when the coefficient of the quadratic term changes from positive to negative. The saddle point solution is obtained by minimizing $\Psi[P]$:

$$P_{sp} = \begin{cases} 2(p - p_c)/3b & , \text{if } p > p_c \\ 0 & , \text{if } p < p_c \end{cases} \qquad (4)$$

We can then calculate the various critical exponents. Near criticality as $p_c \to p_c^+$, $P \propto (p - p_c)^{\beta}$, therefore $\beta = 1$. To get the critical behaviour of the response function, we add to $\Psi[P]$ an external field term $-hP$. This term is the analogy of the magnetic field term in spin models and the response function is the magnetic susceptibility: $\chi \propto |p - p_c|^{-\gamma}$. Minimization gives the solution

$$h = 2(p_c - p)P + 3bP^2, \qquad (5)$$

which leads to

$$\chi^{-1} = \frac{\partial h}{\partial P} \,|_{h=0, P_{sp}} = 6(p - p_c). \qquad (6)$$

thus we obtain $\gamma = 1$. For the divergence of the correlation length, we add fluctuations to the saddle point solution: $P = P_{sp} + \delta P$. Then we have, keeping only the Gaussian part:

$$\Psi[P] = \Psi_{sp} + \left[\frac{K}{2}(\nabla \delta P)^2 + (p_c - p)\delta P^2\right] + O(\delta P^3) \qquad (7)$$

A correlation length can be defined from the Gaussian modes as $\frac{K}{\xi^2} = 2(p_c - p)$ thus

$$\xi \propto |p - p_c|^{-\nu} \qquad where \qquad \nu = \frac{1}{2}. \qquad (8)$$

Now we deal with the DP problem: In the isotropic model rotational symmetry prohibits terms with odd power of $\nabla\delta P \sim \vec{k}\delta\tilde{P}$, where $\delta\tilde{P}$ is the fluctuation in momentum space. For DP, The existence of the forward direction breaks the rotational symmetry and such terms are again allowed. To lowest order, we add only the term $(\vec{a}\cdot\nabla P)P \sim (\vec{a}\cdot\vec{k})P^2$ so that:

$$\Psi_{DP}[P] = \frac{K}{2}(\nabla P)^2 + (p_c - p)P^2 + (\vec{a}\cdot\nabla P)P + bP^3 + \dots \tag{9}$$

here $\vec{a}$ is associated with the forward direction. The exponents γ and β are obtained from the saddle point solution assuming $\nabla P = 0$, thus they remain unchanged for DP: $\gamma = \beta = 1$. For the fluctuations, their contribution to the Hamiltonian now becomes:

$$\int d^d\mathbf{k}\left\{\left[(p_c - p) + \vec{a}\cdot\vec{k} + \frac{K}{2}k^2\right]\delta\tilde{P}^2 + O(\delta\tilde{P}^3)\right\} \tag{10}$$

Keep in mind that we are interested in the long wavelength low energy excitations, i.e. $k \to 0$. Now the linear term distinguishes connectivity in the forward direction from the rest of the spatial directions. For longitudinal modes of $\delta\tilde{P}$ with $\vec{k} \parallel \vec{a}$, the linear term in k dominates over the quadratic term. So the longitudinal correlation length should be written as (from dimensional counting):

$$\frac{a}{\xi_\parallel} = (p_c - p) \qquad \Longrightarrow \qquad \nu_\parallel = 1 \tag{11}$$

In the transverse directions, the linear term vanishes and the modes are Gaussian with the same correlation length as in the isotropic model:

$$\frac{K}{2\xi_\perp^2} = (p_c - p) \qquad \Longrightarrow \qquad \nu_\perp = \frac{1}{2}. \tag{12}$$

From the critical exponents obtained above we can also calculate the upper critical dimension d_u, below which fluctuations render the saddle point solution invalid. The usual Ginzburg criterion reads

$$d_u\nu = 2\beta + \gamma \tag{13}$$

In the isotropic case, $\nu = 1/2$ along all dimensions and we get $d_u = 6$. In the DP problem, two different exponents $\nu_\parallel$ and $\nu_\perp$ exist, and we can modify the Ginzburg criterion as

$$1\cdot\nu_\parallel + (d_u - 1)\nu_\perp = 2\beta + \gamma \tag{14}$$

from which we get for DP, $d_u = 5$.

IV. POSITION SPACE RENORMALIZATION GROUP

Reynolds et al. [8] first developed a position space renormalization group (PSRG) method for isotropic percolation on a square lattice. Later it is generalized for directed percolation on square lattice where as usual the forward direction is along one diagonal[7, 9]. We'd like to first introduce the PSRG method and then use it for the triangular lattice. In DP on square lattice, a $b \times b$ cell is rescaled to a 1×1 cell. A directed bond in the renormalized cell is occupied only if there is a connected path transversing the original cell in that direction. Here we consider the simplest case of $b = 2$, as illustrated in Fig. 4. We'd like to find the renormalized bond occupation probability p'_s in terms of the original probability p_s. The subscript s is used to for Square lattice. Taking contributions from all possible bond configurations in the original lattice, we arrive at the recursion relation:

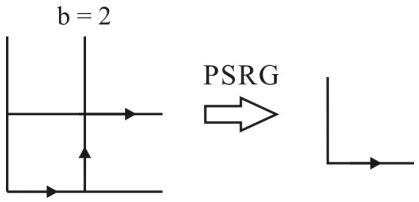

FIG. 4. PSRG: rescaling by factor of 2. There's a connected path in the original cell in the horizontal direction, so the horizontal bond in the renormalized cell is occupied. This particular configuration corresponds to the last term on the RHS of Eqn. 15.

$$\begin{aligned} p'_s &= p_s^4 + 4p_s^3(1-p) + 2p_s^2(1-p_s)^2 + p_s^3(1-p_s)^2 \\ &= p_s^5 - 3p_s^4 + p_s^3 + 2p_s^2. \end{aligned} \tag{15}$$

Solve for $p'_s = p_s$, we obtain the fixed points: $p_s^* = 0, 0.555,$ and 1. The stable point points at 0 and 1 reassures that p'_s is a properly defined probability. The unstable fixed point $p_s^* = 0.555 = p_s^c$ is the critical point of the square lattice DP phase transition. Linearize the recursion relation near the critical value gives $\delta p\prime = 1.567\times\delta p$. Compare this to the definition of the critical exponent in RG: $\delta p' = b^{y_p}\delta p$, we obtain $y_p = \ln 1.567/\ln 2 = 0.648$. Then the scaling relations in RG gives the exponent of the correlation length as $\nu = 1/y_p = 1.54$.

To apply this PSRG scheme to the triangular model, we make use of the previous observation that the triangular DP we consider can be viewed as a square DP with additional transverse bonds. We want to integrate out the transverse bonds by absorbing their contribution to forward connectivity into the occupation probability of the longitudinal bonds.

Refer back to Fig. 2, consider the sites $s_i(t)$ and $s_i(t+1)$. For the square DP, they can only be connected by the bond in between them with a probability p_s. However in the triangular lattice, the two sites can also be connected via other paths which utilizes the transverse bonds. To lowest order, the connected paths $s_i(t) \to s_{i+1}(t) \to s_i(t+1)$ and $s_i(t) \to s_{i-1}(t+1) \to s_i(t+1)$ appear each with probability $p\cdot(p/2)$. Therefore, we have:

$$p_s = p + 2\times p\cdot(p/2) + O(p^3) \tag{16}$$

With the previous result $p_s^c = 0.555$, we can use Eqn. 16 to solve the critical value of the triangular DP as $p_c = 0.397$. Higher order terms can be calculated by including paths that travel further along the transverse direction. The resulting p_c will converge from above as order terms are terms are added in Eqn. 16.

Using the lowest order result, the linearization of the recursion relation around p_c is identical to that for p_c, yielding the same exponent $\nu = 1.54$. However for the triangular DP, since we are only considering forward connectivity in the PSRG, this exponent should be interpreted as $\nu_\parallel$.

V. CONCLUSION

We have looked into directed bond percolation problem on a 2D triangular lattice. We have developed a dynamic algorithm to formulate the DP problem as a 1D time-evolution problem. A simulation is carried out using this algorithm to give an approximate value for p_c. We used an mean field approach to calculate the critical exponents for the DP phase transition: $\beta = \gamma = 1$, $\nu_\parallel = 1$ and $\nu_\perp = 1/2$. A upper critical dimension of $d_u = 5$ is obtained using the Ginzburg criterion. We also developed a position space renormalization group scheme to investigate the critical point and get a critical value $p_c = 0.397$ and the exponent of the longitudinal correlation length $\nu = 1.54$.

Remarks: It is worth noting that despite of all the extensive theoretical work carried out on DP models, there have been few experiments where the critical behaviour of the DP phase transition is directly measured, mainly due to obstructions from fluctuations or quenched disorder[4]. Lots of open questions remain[10], making the field still fascinating and full of challenges.

[1] D. Stauffer, Phys. Rep., **54**, 1 (1979).
[2] J. W. Essam, Rep. Prog. Phys., **43**, 833 (1980).
[3] S. R. Broadbent and J. M. Hammersley, Proc. Camb. Phil. Soc., **53**, 629 (1957).
[4] H. Hinrichsen, Adv. Phys., **49**, 815 (2000).
[5] D. Mollison, J. Roy. Stat. Soc. B, **39**, 283 (1977).
[6] E. V. Albano, J. Phys. A, **27**, L881 (1994).
[7] S. Redner, Phys. Rev. B, **25**, 3242 (1981).
[8] W. K. P. J. Reynolds and H. E. Stanley, J. Phys. C, **10**, L167 (1977).
[9] P. J. R. H. Nakanishi and S. Redner, J. Phys. A: Math. Gen., **14**, 855 (1981).
[10] P. Grassberger, Proc. 1995 Shilma Conf. on Complex Systems (1997).