Berezinskii-Thouless-Kosterlitz Transition in a Two-Dimensional Atomic Gas

Jee Woo Park
Department of Physics, MIT
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We introduce the concepts of Berezinskii-Kosterlitz-Thouless (BKT) transition and quasi long-range order in the context of the 2D electron gas model. We then apply this framework to 2D superfluidity to obtain the superfluid transition temperature. Finally, we review the recent experimental work of Hadzibabic *et al.* which observes BKT transition in a 2D trapped ^{87}Rb gas. Thermal de Broglie wavelength at the critical temperature is reported to be $\rho_C \lambda^2 = 6 \pm 2$, in comparison to the theoretical result of $\rho_S \lambda^2 = 4$, where ρ_C is the condensate density and ρ_S is the superfluid density.

INTRODUCTION

In 1966, Mermin and Wagner presented a rigorous argument that for a system of $d \leq 2$ spatial dimensions with continuous symmetry, thermal fluctuations of the system inhibits formation of long-range order at any finite temperature [1]. However, motivated by Berezinskii's theoretical work on the low temperature behavior of a 2D superfluid [2, 3], Kosterlitz and Thouless introduced the thermodynamic model of topological order in 1973 to account for the quasi long-range order present in low temperature states of two dimensional systems [4]. The transition to the quasi ordered phase in two dimensions is called the Berezinskii-Kosterlitz-Thouless (BKT) transition, and it remains to be a topic of heavy interest.

Particularly, many expereiments have attempted to observe the BKT transition and study its physics in various physical contexts [5–7]. One lacking of these efforts is that they are limited to studying the transition indirectly through macroscopic variables, due to the nature of the experiments. However, in 2006, Hadzibabic *et al.* carried out a noteworthy experiment using cold atoms to directly observe the phase correlation of a cloud of trapped ^{87}Rb gas going through a BKT transition into a superfluid phase.

In this paper, we review the experimental work of Hadzibabic *et al.* in the context of the theoretical model laid out by Kosterlitz and Thouless. An estimate of the critical temperature will be given and compared to the experimental result. Possible future experimental work will be presented in the end.

BKT THEORY OF QUASI LONG-RANGE ORDER

Topological Defects

In two dimensions, low energy thermal fluctuations (i.e. Goldstone modes) destroy true long-range order of any system with continuous symmetry. However, a new kind of long range order without explicit symmetry breaking may be achieved by introducing the notion of topological order.

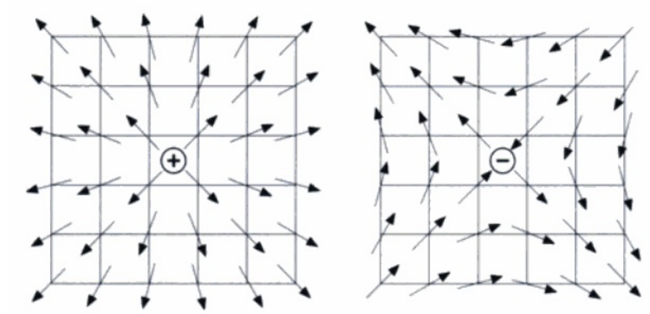

FIG. 1: Topological defects in the 2D XY model [8]

In the low temperature phase of a two dimensional system, one can imagine having singular deformations of the field describing the system at hand (e.g. spin field for magnetic systems) as in FIG. 1 which cannot be characterized by the usual gradual deformations caused by the Goldstone modes. We call these deformations topological defects, and attribute the disordering of the quasi ordered-phase of the system to the proliferation of these defects.

For each of the defects, we define the topological charge to be the winding number of the field describing the system as we loop around the defect in position space. At low temperatures, defects of equal magnitude and opposite sign can pair up to form a dipole since the field deformation caused by one charge will nearly be cancelled by the other charge with opposite sign, decreasing the overall energy cost of maintaining these defects. As the temperature of the system is raised, the size of the dipole grows and eventually, past a certain critical temperature, defects will behave as free entities forming a plasma of charged defects.

The theory of topological defects is reminiscent of that of a 2D Coulomb gas, and indeed the energy cost of field deformation due to topological defects will turn out to have a Coulomb law like behavior. With this foresight, the theory of topological defects will be developed in the context of a 2D Coulomb gas.

Model System: 2D Coulomb Gas

Consider a gas of charged particles with charge $\pm q$ in two dimensions. In 2D, the Coulomb potential is logarithmic, and the interaction energy of a pair of charges can be written as

$$U(|\vec{r_i}-\vec{r_j}|) = \begin{cases} -2q_iq_j \ln\left|\frac{\vec{r_i}-\vec{r_j}}{r_o}\right| + 2\mu & (\text{for} \quad r > r_o) \\ 0 & (\text{for} \quad r < r_o) \end{cases} \tag{1}$$

where we define a lower length cutoff at r_o to avoid divergences at small separations. r_o may be thought of as the size of a charge. μ is the energy cost of creating a charge, which is the same for $+$ and $-$ charges.

In the low temperature limit, electrons will form pairs of tightly bound dipoles, where as in the high temperature limit, size of the dipole pairs will grow until the gas becomes a plasma of charged particles. A simple way to estimate the the critical temperature of this transition is to consider the free energy F of an isolated charge. Considering contributions from both energy and entropy,

$$\begin{aligned} F &= E - TS \\ &\cong \frac{1}{2}(2q^2 \ln \frac{L}{r_o}) - T(k_B \ln \frac{L^2}{r_o^2}) \\ &= (\frac{1}{2}q^2 - k_BT) \ln \frac{L^2}{r_o^2} \end{aligned} \tag{2}$$

and we see that $F \to 0$ as $k_BT \to q^2/2$. At $k_BT_c = q^2/2$, the cost of liberating a free charge is zero, and there will be a proliferation of free charges. A more robust estimate of T_c can be given using a renormalization group treatment which takes into account possible interactions between dipoles. This approach will be described in the following section.

Defects in the 2D XY Model and 2D Superfluidity

On his seminal paper [2], Berezinskii identifies a correspondence between the 2D XY model and 2D superfluidity, so it is natural to develop the theory in the context of the 2D XY model. Instead of the discrete lattice picture, we take a mean field approach and define the distortion field $\vec{u}$ of the spin field from the perfectly aligned ground state as

$$\vec{u} = \nabla\phi + \nabla \times (\hat{z}\psi) \tag{3}$$

where $\nabla\phi$ represents continuous deformations of the spin field (i.e. Goldstone modes), and $\nabla \times \hat{z}\psi$ accounts for singular deformations of the field (i.e. vortices). Introducing a stiffness constant K, the energy cost of the deformation can be calculated as

$$\begin{aligned} \beta H &= \frac{K}{2}\int d^2\mathbf{x}\, |\vec{u}|^2 \\ &= \frac{K}{2}\int d^2\mathbf{x} (\nabla\phi)^2 - \int d^2\mathbf{x}\psi\nabla^2\psi \end{aligned} \tag{4}$$

after integrating by parts. Applying Stokes' theorem around the topological defects, we obtain

$$\begin{aligned} \nabla \times \vec{u} &= \hat{z}(\nabla^2\psi) \\ &= 2\pi\hat{z}\sum_i n_i \delta^2(\mathbf{x}-\mathbf{x}_i) \end{aligned} \tag{5}$$

where n_i is the topological charge of a defect positioned at $\mathbf{x}_i$. Solving this equation yields $\psi(\mathbf{x}) = \sum_i n_i \ln(\frac{|\mathbf{x}-\mathbf{x}_i|}{a})$ and from EQN. 4 we obtain the topological defect contribution to the energy as

$$\beta H_t = -\pi K \sum_{i,j} n_i n_j \ln\left|\frac{\mathbf{x}_i - \mathbf{x}_j}{a}\right| \tag{6}$$

$$= \begin{cases} -\pi K \sum_{i\neq j} n_i n_j \ln\left|\frac{\mathbf{x}_i-\mathbf{x}_j}{a}\right| - \pi K \sum_i n_i^2 \ln(\frac{r_o}{a}) \\ \qquad (\text{for} \ \ |\mathbf{x}_i - \mathbf{x}_j| > a) \\ 0 \qquad (\text{for} \ \ |\mathbf{x}_i - \mathbf{x}_j| < a) \end{cases} \tag{7}$$

where r_o is roughly the size of a defect and a is a lower length cutoff on the order of the lattice spacing.

Near the critical temperature T_c, tightly bound dipoles will partially screen the interaction between longer dipoles, and by computing the effective interaction perturbatively to lowest order of y_o, we obtain an effective stiffness constant [8]

$$K_{\text{eff}} = K - 4\pi^3K^2y_o^2a^{2\pi K}\int_a^\infty dr\ r^{3-2\pi K} \tag{8}$$

By constructing a renormalization group for the effective interaction with a scaling factor $b = e^l$, we obtain the following recursion equations [8]

$$\begin{cases} \frac{dK^{-1}}{dl} = 4\pi^3a^4y_o^2 \\ \frac{dy_o}{dl} = (2-\pi K)y_o \end{cases} \tag{9}$$

Note that there is an abrupt change of flow at $K_c^{-1} = \pi/2$. For $K^{-1} < K_c^{-1}$ and small y_o, the flow terminates on a fixed line, which implies that only dipoles of finite size exist. However, for $K^{-1} > K_c^{-1}$, the flow diverges, which implies that perturbation theory breaks down due to proliferation of vortices.

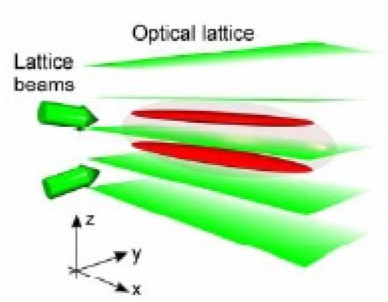

FIG. 2: 1D optical lattice to cut and squeeze the cloud in the z direction [9]

Using Berezinskii's correspondece between the 2D XY model and a 2D superfluid [2], we obtain $K \leftrightarrow \hbar^2\rho_S/2m^*k_BT$, where ρ_S is the superfluid density and m^* is the effective mass of the superfluid. Hence, from $K_c^{-1} = T_c = \pi/2$, we estimate the superfluid transition temperature T_c to be

$$\frac{2}{\pi} = \frac{\hbar^2\rho_S}{mk_BT_c} \Leftrightarrow \Lambda^2\rho_S = 4 \tag{10}$$

where Λ is the thermal de Broglie wavelength $\Lambda \equiv \sqrt{2\pi\hbar^2/k_BT_cm^*}$.

EXPERIMENTAL PROCEDURE AND RESULTS

The first step of the experiment is to prepare a three dimensional quantum degenerate Bose gas of ^{87}Rb. This is done by a standard procedure of loading a pre-cooled cloud of ^{87}Rb atoms into a cylindrically symmetric magnetic trap, and then further cooling the cloud to the desired temperature through evaporative cooling. Evaporative cooling is a technique where radio-frequency (RF) radiation is applied to the trapped atomic cloud in order to remove hot atoms which lie on the tail of the Boltzmann velocity distribution. RF radiation will induce a transition of the spin state of hot atoms to high-(magnetic) field seeking states which can leave the trap, while the remaining atoms rethermalize.

Next, by having two laser beams interfere and form a standing wave on top of the atomic cloud, a one dimensional optical lattice is created which cuts the degenerate gas into two, as shown on FIG. 2. The lattice potential is then gradually increased to compress the two clouds into two dimensional surfaces parallel to the $x-y$ plane. It is important that the ramp rate be slow in order to minimize heating of the clouds in the z direction. The reported lattice spacing is 3μm and the ramp time is 500ms, with an additional equilibrium time of 200ms applied afterwards.

The crucial idea behind directly observing phase correlation in these 2D atomic clouds is to make use of matter-wave interference. If there exists (quasi) long-range order within the two clouds, when all trapping fields and lattice beams are turned off and the two clouds are allowed to expand, matter-wave interference pattern of the expanding clouds would only have long wavelength fluctuations. Based on the BKT theory, as the temperature of the clouds is raised, we expect to see a sharp transition from the quasi ordered state to the disordered state, where free vortices of the cloud wavefunction proliferates and long range order is destroyed.

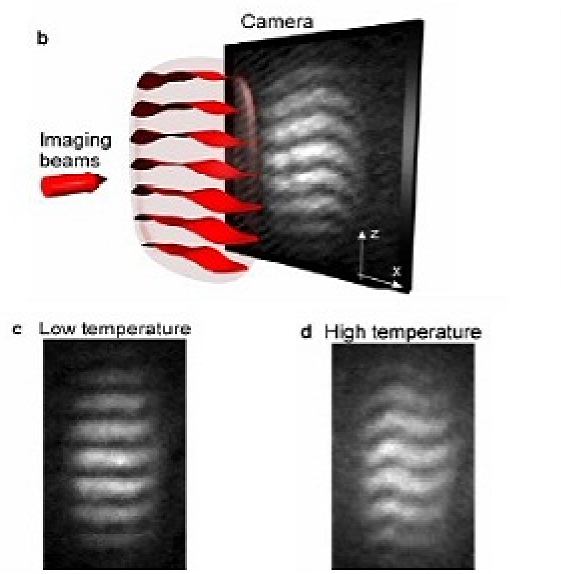

FIG. 3: Interference pattern of the atomic clouds after TOF = 20ms. [9]

With this in mind, all of the potentials are turned off, and the 2D clouds are allowed to expand in the z direction for 20ms, after which an image of the clouds in the $x-z$ plane is taken using resonant light. Locations of constructive matter-wave interference will appear as dark strips due to absorption. Two images, one taken at low temperature and another taken at high temperature, are reported on FIG. 3. Indeed, a straight interference pattern characteristic of long range order is observed on the low temperature image, compared to the wavy pattern of the high temperature image.

For a quantitative analysis of the data, one can fit a profile of the form $F(x,y) = G(x,z)[1+c(x)\cos(2\pi z/D + \phi(x))]$ to the image, where $G(x)$ is a Gaussian envelope that naturally arises from the density profile of the atomic cloud, $c(x)$ is an amplitude term which captures the local coherence, and $\phi(x)$ is a phase factor which captures the long range coherence. A. Polkovnikov, E. Altman, and E. Demler [10] have proposed a theoretical model which relates the spatial correlation of the cloud wavefunction $g_1(\vec{r},\vec{r\prime}) = \left\langle \psi(\vec{r})\psi(\vec{r\prime})\right\rangle$, where $\psi(\vec{r})$ is the fluctuating bosonic field at location $\vec{r}$, to the length of the long axis of the two dimensional cloud:

$$\langle C^2(L_x)\rangle \approx \frac{1}{L_x}\int_0^{L_x}[g_1(x,0)]^2dx \propto (\frac{1}{L_x})^{2\alpha} \tag{11}$$

Note that in this analysis, long range correlation is captured in a single exponent, α. If there is perfect coherence of the phases within the two clouds, one would expect to see a straight line pattern on the image, parallel to the x axis. Then, $g_1(x,0)$ would be constant and $\alpha = 1/2$. Loss of coherence leads to decay of the correlation function, which then pushes the exponent down. For the case at hand, it can be calculated that $\alpha = 1/2$ for the low temperature phase and $\alpha = 1/4$ for the high temperature phase.

Based on this analysis, the experiment measures

$$\tilde{C}(L_x) = \frac{1}{L_x}\left|\int_{-L_x/2}^{L_x/2} c(x)e^{i\phi(x)}\right| \tag{12}$$

from the fitted $F(x,y)$. The values of α obtained from $\tilde{C}(L_x)$ averaged over many images are reported on FIG. 4. Note that there is a crossover region where α gradually changes from 0.50 to 0.25, in agreement with the theoretical model. The temperature of the cloud is inferred using the time-of-flight (TOF) technique, where the wings of the spatial distribution of the cloud after a certain TOF is fitted to an exponential decay function which depends on temperature [11]. The critical temperature is reported to be $T_c = 290 \pm 40$nK, and the peak condensate density is reported to be $\rho_C = (5 \pm 1) \times 10^9\text{cm}^{-2}$, yielding $\rho_C\lambda^2 = 6 \pm 2$. This result is within the theoretical value of $\rho_S\lambda^2 = 4$.

In addition to the above result, direct observation of free vortices is reported. In contrast to paired dipoles, free vortices would appear on the image as abrupt dislocations on the image at temperatures higher than T_c. As shown on FIG. 5, the number of images containing dislocations increases with a sudden jump past T_c. The coincidence of the temperature at which the dislocations proliferate and T_c strongly supports the existance of quasi long-range order below T_c.

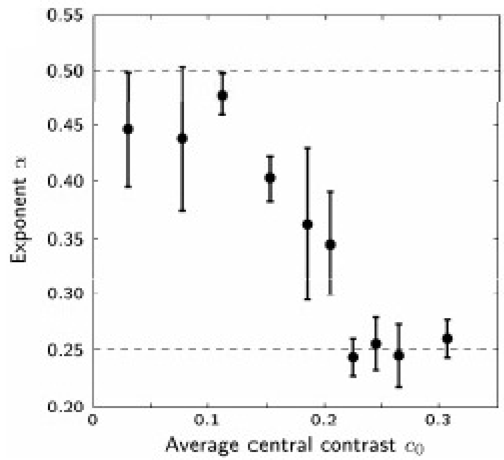

FIG. 4: BKT crossover in a 2D gas. Temperature is reported in terms of c_0, the average local contrast, which can be directly related to the condensate temperature. [9]

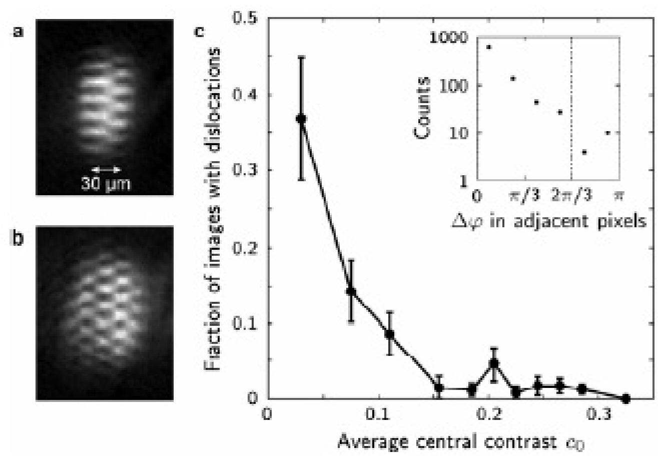

FIG. 5: Dislocations caused by free vortices. Lighter central contrast (i.e. smaller c_o) corresponds to higher temperature. [9]

Future Works

Although the results presented above are impressive, some questions remain to be further addressed. First of all, whether the 2D gas truly carries the characteristics of a superfluid is not answered. Superfluidity in quantum degenerate gases can be observed by stirring the atomic cloud and directly observing the emergence of quantized vortex lattices [12]. Also, due to the interaction between constituent particles, a two dimensional cloud of ^{87}Rb is expected to have a Bose-Einstein condensate phase at sufficiently low temperatures [9]. Observation of the BEC phase would further solidify the agreement between theory and the results reported by Hadzibabic *et al.*

CONCLUSION

In this paper, the theory of Berezinskii-Kosterlitz-Thouless (BKT) transition is presented through a simplified model of 2D electron gas. This model is then mapped to the picture of 2D superfluidity to calculate the critical temperature T_c at the onset of superfluidity. Then, the experimental work of Hadzibabic *et at.* is reviewed in context of the model presented. The reported T_c given by $\rho_C \lambda^2 = 6 \pm 2$ is within the theoretical estimate of $\rho_S \lambda^2 = 4$. This result is further confirmed by direct observation of free vortices above T_c. Directions for further investigations include direct observation of superfluidity in the quasi ordered phase and the observation of Bose-Einstein condensation at even lower temperatures below T_c.

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