Examining the Effects of Fluctuations on the Critical Casimir Force

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In this paper we examine both the standard Casimir force and, more relevant to this class, the critical Casimir force which arises from fluctuations of a fluid near the λ transition. We find that adding fluctuations leads to a reduction in amplitude of the mean field force which is consistent with experimental results.

INTRODUCTION TO THE CASIMIR FORCES

Confinement of vacuum fluctuations which result in macroscopic forces between neutral objects is known as the Casimir force [1, 2]. This force was predicted by Hendrik Casimir in 1948, and has been studied experimentally in various geometries consisting of neutral parallel plates or similar configurations [3]. In geometries involving vacuum-separated objects, the Casimir force is always attractive and decays as a function of object separation [2]. However, in more complicated geometries the Casimir force has been shown to exhibit non-monotonic behavior and in some cases establish a stable equilibrium between objects [4].

Because the Casimir force arises from the confinement of fluctuations of the electro-magnetic field it is not unreasonable to expect Casimir-like effective forces to arise from the confinement of a fluctuating medium near phase transition. Furthermore, near the phase transition the fluctuations of the order parameter occur on all length scales and the critical Casimir force should exhibit universality [5]. Experimental tests on thin super-fluid films have shown that such forces exist and they obey the form predicted by universal scaling [6, 7], namely:

$$\frac{f(T,L)}{k_B T_c} = \frac{1}{L^d}\Theta\left(rL^{1/\nu}\right) \tag{1}$$

Where $f(T,L)$ is the Casimir force per unit area, Θ is a universal scaling function, $r = (T - T_c)/T_c$ is the reduced temperature, L is the thickness of the film, and ν is defined by $\xi \propto |r|^{-\nu}$. In this paper we discuss the electro-magnetic Casimir force and the critical Casimir force in simple geometries depicted in Fig. 1 and Fig. 2.

THE STANDARD CASIMIR EFFECT

In this section we discuss the standard Casimir effect. While there is little correlation to the material covered in class, this section is designed to demonstrate a simple calculation of the Casimir force for two parallel-conducting plates in order to compare with the calculation of the critical Casimir force in the next section. To begin the derivation we start off with the energy of the electromagnetic field, namely:

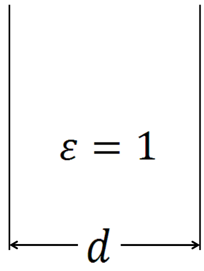

FIG. 1: Geometric configuration for the calculation of the electro-magnetic Casimir force. At z=0 and z=d there is two perfectly conducting parallel plates

$$E = \sum_n \frac{\hbar\omega_n}{2} \tag{2}$$

Where ω_N are the allowed frequencies of our geometry (see Fig. 1). Upon considering the allowed modes and subtracting the infinite contribution to the energy we can write Eq. (2) as:

$$E(d) = \hbar c \frac{L^2}{\pi^2}\frac{\pi}{2}\Bigg[\sum_n \int_0^\infty dk k\sqrt{k^2 + \frac{\pi^2 n^2}{d^2}} - \frac{d}{\pi}\int_0^\infty dk_z \int_0^\infty dk k\sqrt{k^2 + k_z^2}\Bigg] \tag{3}$$

Where $L \times L$ is the area of the plate, and $k = \sqrt{k_x^2 + k_y^2}$. We then employ the use of a regulator function, $f(\sqrt{k^2 + \frac{\pi^2 n^2}{L_z^2}}) = f(x)$ such that $f(x) = 0$ for $x \gg x'$ and $f(x) = 1$ for $x \ll x'$. Here $x' \approx 1/a$ where a is the average atomic separation of the conductor [8]. The physical interpretation of the regulator function is that no material is a "true" conductor, and materials are transparent to light whose wavelength is smaller than the atomic spacing.

Applying the regulator to Eq. (3) and introducing the function $G(x)$ as:

$$G(x) = \int_0^\infty du\sqrt{u+x^2} f\left(\frac{\pi}{d}\sqrt{u+x^2}\right) \tag{4}$$

we can rewrite Eq. (3) as:

$$E(d) = \frac{\pi^2 \hbar c L^2}{4d^3}\left(\frac{1}{2}G(0) + \sum_{n=1}^{\infty} G(n) - \int_0^\infty ds G(s)\right) \tag{5}$$

Where we used the change of variables $x = k^2d^2/\pi^2$ and $s = k_z d/\pi$. The 1/2 in front of $G(0)$ arises from the fact that we multiplied every ω contribution by 2 since there are two polarizations of light, however if $n = 0$ this is no longer the case, this is the reason for the 1/2. Employing the Euler-Maclaurin summation (see [9]) as is done in [8] we find the energy can be written as:

$$E(d) = -\frac{\pi^2 \hbar c}{720 d^3} L^2 \tag{6}$$

Eq. (6) is the Casimir energy for two parallel perfectly conducting plates. Upon taking the derivative and dividing by L^2 we find that the force per unit area is given by:

$$F(d) = -\frac{\partial E}{\partial d} = -\frac{\pi^2 \hbar c}{240 d^4} \tag{7}$$

To close this section I would like to comment on the derivation of Eq. (7). The main task was to find the energy configuration of the system, Eq. (6). After finding the energy the force was obtained by taking its derivative. This procedure, finding the energy and then taking the derivative, will be similar to what we will do to find the critical Casimir force.

CRITICAL CASIMIR FORCE

We now consider the configuration depicted in Fig. 2. Where we have a super-fluid ^4He film of thickness L, surrounded by vapor and suspended above bulk ^4He liquid. This configuration describes experiments done by [10]. The first step in finding the Casimir force is to find an energy whose derivative will yield the critical Casimir force. The energy needed in this case is the free energy $\mathcal{F}$. To find the free energy we construct the system's Landau Ginzburg Hamiltonian. Considering the symmetries of the problem and limiting ourselves to one dimension(moving through the film $d = 1$), we have the following Landau-Ginzburg Hamiltonian [6].

$$\beta\mathcal{H} = \int_0^L dz\left[\frac{K}{2}\left(\frac{d\psi}{dz}\right)^2 + \frac{r}{2}\psi(z)^2 + u\psi(z)^4\right] \tag{8}$$

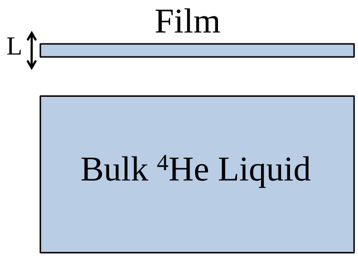

FIG. 2: Configuration for the calculation of the critical Casimir force. We have a super-fluid thin film suspended above bulk ^4He fluid

Where u is the fourth order coupling constant, and $\psi(z)$ is the order parameter which must satisfy $\psi(0) = \psi(L) = 0$ due to the physical constraints of the system. Upon taking the functional derivative of the Hamiltonian we get:

$$0 = -K\frac{d^2\psi}{dz} + r\psi^2 + u\psi^4 \tag{9}$$

In the limit $L \to \infty$ the solution to Eq. (9) is given by a tanh function. However for general L the saddle point solution which satisfies Eq. (9) for $r < 0$ is given by a Jacobi elliptic function [12]. However, finding the correct function which satisfies the boundary conditions is a difficult task and finding a solution may not be of any use. Therefore, in order to obtain feasible result, we replace $\psi(z)$ by it's maximum value ψ_0 (as of yet unknown) and assume ψ constant. Once we do the replacement and take the derivative, as was done for the standard Casimir force, we obtain:

$$\frac{\partial \mathcal{F}}{\partial L} = \frac{r}{2}\psi_0^2 + u\psi_0^4 \tag{10}$$

Now our only task is to evaluate ψ_0. To do this we first multiply equation Eq. (9) by $d\psi/dz$ and integrate from $z = L/2$ to some z. By symmetry $\psi'(L/2) = 0$ and choosing units such that $K = 1$ we obtain.

$$\frac{1}{2}\left(\frac{d\psi}{dz}\right)^2 = \frac{r}{2}\left(\psi(z)^2 - \psi_0^2\right) + u\left(\psi(z)^4 - \psi_0^4\right). \tag{11}$$

Eq. (11) is a separable differential equation which can be solved for z. Letting $z = L/2$ we obtain.

$$\frac{L}{2} = \int_0^{\psi_0} \frac{d\psi}{\sqrt{r\left(\psi^2 - \psi_0^2\right) + 2u\left(\psi^4 - \psi_0^4\right)}} \tag{12}$$

Using the change of variable $\psi = \psi_0\gamma$ and defining

$\zeta = \frac{2u\psi_0^2}{-r}$ we can write Eq. (12) as;

$$\frac{L}{2} = \int_0^1 \frac{d\gamma}{\sqrt{-r\left(1-\gamma\right)\left(1-\zeta-\zeta\gamma^2\right)}}$$
$$= \frac{1}{\sqrt{-r\left(1-\zeta\right)}} \int_0^1 \frac{d\gamma}{\sqrt{\left(1-\gamma^2\right)\left(1-\frac{\zeta}{1-\zeta}\gamma^2\right)}} \tag{13}$$

We then use the definition of K,the elliptic integral of the first kind, to write Eq. (13) as;

$$\frac{\sqrt{-r}L}{2} = \frac{K\left(\frac{\zeta}{1-\zeta}\right)}{\sqrt{1-\zeta}} \tag{14}$$

In the next section we discuss the series expansion for ψ_0 but for now it is enough to know that Eq. (14) tells us the solution of ζ can be written as a function of $x = \sqrt{-r}L/2$. The derivative of the free energy is then given by.

$$\frac{\partial \mathcal{F}}{\partial L} = \frac{-r^2}{4u}\zeta(x)\left(1-\zeta(x)\right)$$
$$= \frac{-4x^4}{uL^4}\zeta(x)\left(1-\zeta(x)\right) \tag{15}$$

To obtain a final expression for the critical Casimir force we must take into account that the experiment of the super-fluid thin films were done in vapor [5, 10]. Therefore we must take into account the difference in free energy of removing some ^4He liquid from the film and adding it to the bulk. The free energy of the bulk can easily be found using saddle point approximation and is given by $\mathcal{F}_{bulk} = -r^2L/16u$. Upon taking the derivative of F_{bulk} the expression for the force becomes;

$$F = \frac{\partial}{\partial L}\left(\mathcal{F}_{bulk} - \mathcal{F}\right)$$
$$= \frac{-x^4}{L^4u} + \frac{4x^4}{uL^4}\zeta(x)\left(1-\zeta(x)\right) \tag{16}$$

Where F is the critical Casimir force. Eq. (16) is the mean field estimate of the critical Casimir force. In the following sections we will examine this formula and the modifying effects of fluctuations.

SERIES EXPANSION OF THE FORCE

For completeness we desire to give Eq. (16) as a power series instead of an expression of some unknown function $\zeta(x)$. First we consider expanding about $r \approx 0$. However, we know that for $\sqrt{-r}L < \pi$ there is no solution for ζ. This can be seen in Fig. 3, which plots the left hand side of Eq. (14). From the figure we see that if $\sqrt{-r}L < \pi$ then $\zeta < 0$ which goes against our assumptions. Therefore the expansion for small r is trivial, it is just the bulk free energy term, and we will only consider an expansion for $-rL > \pi$. To find an expansion for the force we must first find and expansion for ζ. To do this we have to go back to Eq. (14) and take the inverse of both sides. We take the inverse of the equation because we wish to expand ζ in terms of $1/\sqrt{-r}$ instead of r. Once we take the inverse, we expand the right hand side of Eq. (14) in power series of ζ. To expand the elliptic integral we use the following identity given in [9];

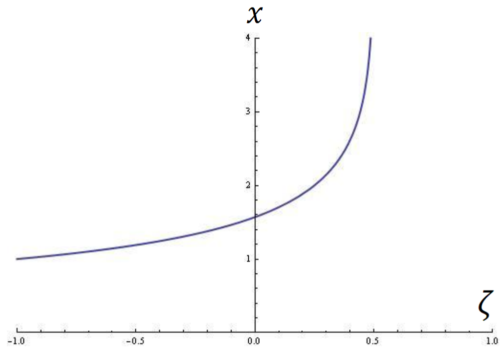

FIG. 3: Plot of Eq. (14) which shows for $\sqrt{-r}L < \pi$ there exists no solution to Eq. (9)

$$K(\kappa) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}n!^2}\right]^2 k^{2n} \tag{17}$$

Once the right hand side of Eq. (14) is a power series in ζ we can use the method of reversion of power series [9] to solve for ζ. The first few terms of ζ are then;

$$\zeta(x) \approx -\frac{4}{3}\lambda - \frac{7}{9}\lambda^2 + \frac{5}{54}\lambda^3 + ... \tag{18}$$

Where we define

$$\lambda = \frac{\pi}{2x} - 1 \tag{19}$$

TABLE I: Coefficients in a power series of the form $\frac{L^4uF}{x^4} = \sum c_n\lambda^n$

Order	Coefficient(c^n)
0	-1
1	-5.3333
2	-10.2222
3	-7.92593
4	-1.7963
5	-0.097736
6	0.0242198
7	0.0665152

Due to the constraints on r and L, λ is constrained as $-1 < \lambda < 0$. This constraint on lambda is why we chose it to be the expansion parameter of the force.

Once we have an expansion for ζ the expansion for the force is trivial. Table 1 lists the first seven coefficients of the expansion in lambda of the critical Casimir force. By means of the methods described above the series for the force can be computed to any accuracy. The only drawbacks is that the series converges slower for larger values of r and converges the fastest when $x \approx \pi/2$. Furthermore, the series is alternating and better approximations may be found by using Pade Approximants.

ADDING FLUCTUATIONS

After discussing the saddle point approximation to the critical Casimir force we wish to know the effect of fluctuations on the force. We consider the case which the scalar field is written as,

$$\psi = \psi_0 + a(x) \tag{20}$$

Where $a(x)$ is a fluctuation field. Substituting Eq. (20) into the Landau-Ginzburg Hamiltonian and keeping terms up to quadratic order we obtain;

$$\begin{aligned}\beta\mathcal{H} = \beta\mathcal{H}_0 + \int_0^L dz \left(\frac{1}{2}\left(\frac{da}{dz}\right)^2 + r + 12u\psi_0^2 \right) \\ = \beta\mathcal{H}_0 + \beta\mathcal{H}'\end{aligned} \tag{21}$$

Where $\mathcal{H}_0$ is the Hamiltonian free of fluctuations. Going to Fourier modes we can diagonalize the Hamiltonian to get;

$$\beta\mathcal{H}' = \int \frac{dq}{(2\pi)} \left(q^2 + r + 12u\psi_0^2\right) |a|^2 \tag{22}$$

The partition function then decouples into a product of gaussians and we can use the λ expansion developed in the previous section to write the fluctuation contribution to the free energy as.

$$\mathcal{F}' \approx \frac{k_b T L}{2} \int \frac{dq}{(2\pi)} \log r \left(\frac{1}{r} q^2 + 1 + 8\lambda \right) \tag{23}$$

Where we approximated ψ_0 only to order λ and we took L to be large with respect to the lattice spacing. Once we take the difference in energy between the bulk and the film we can take a derivative to obtain the critical Casimir force due to fluctuations. This force, F', is given by;

$$F' = \frac{k_b T}{2} \int \frac{dq}{(2\pi)} \log \frac{\frac{q^2}{-r} + 2}{\frac{q^2}{-r} - (1 + 8\lambda)} \tag{24}$$

In the calculation of F' we assumed that the number of q modes was the same for both the bulk and the film. This is clearly not the case. Therefore, F' serves as a lower bound to the true force from fluctuations. Much information can be gained if we examine the functional form of F' within the λ region in which our series expansion is most valid, $\lambda \approx 0$. In this λ region we see that fluctuations add a positive contribution to the saddle point result, thereby reducing the negative amplitude of the force. This reduction in amplitude is expected since the saddle point amplitude is approximately five times larger than experimental results [6]. Furthermore, even at the lowest order we see that there exist a $\lambda_c = -3/8$ such that $\lambda < \lambda_c$ will result in negative force contributions from fluctuations. This negative contribution for large λ (equivalently large$\sqrt{-r}L$) is also consistent with experimental results [6]. However, due to the slower convergence for large λ we require further terms in the expansion of ζ to obtain the correct behavior for large λ.

CONCLUSION AND POSSIBLE THEORETICAL TREATMENTS

In this paper we have derived both the E-M and critical Casimir force for parallel-plate like geometries. We have seen that the basic steps of calculating these seemingly different forces is the same. The steps involve finding the energy of the system and then taking it's derivative. However, there is a problem in our calculation of the critical Casimir force. In the previous treatment of the critical Casimir force we used a one component scalar field as the order parameter of the super-fluid thin film. Due to the fact that a super fluid is accurately described by a two component order parameter [11] we expect the following Landau Ginzburg Hamiltonian to produce better results.

$$\begin{aligned}\beta\mathcal{H} = \int_0^L dz \Bigg[\frac{K}{2} \left(\left(\frac{d\psi_R}{dz}\right)^2 + \left(\frac{d\psi_I}{dz}\right)^2 \right) \\ + \frac{r}{2}\left(\psi_R^2 + i\psi_I^2\right) + u\left(\psi_R^4 + \psi_I^4 + 2\psi_R^2\psi_I^2\right) + ... \Bigg]\end{aligned} \tag{25}$$

Where we have defined the super-fluid order parameter as $\psi = \psi_R + i\psi_I$. Upon taking the functional derivative of the landau ginzburg Hamiltonian we obtain the following equations for the most probable field configuration.

$$0 = -\frac{d^2\psi_R}{dz^2} + r\psi_R + 4u\left(\psi_R^3 + \psi_R\psi_I^2\right) \tag{26}$$

$$0 = -\frac{d^2\psi_I}{dz^2} + r\psi_I + 4u\left(\psi_I^3 + \psi_R^2\psi_I\right) \tag{27}$$

The next step would be to solve Eq. (26) and Eq. (27) for the maxima of ψ_R and ψ_I. If there was no coupling terms of the form $\psi_i^2\psi_j$ then the solution would be similar to what was shown in the critical Casimir section. Unfortunately these coupling terms have alluded a closed analytical solution and further work must be done to obtain an approximate form of the solution.

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